Solutions FHLF01 FHL064 2012-05-31

Task 1

The temperature T is discretized as

$$\text{T(x,y,t)} = \mathbf{N}(x,y)\mathbf{a}(t) = \sum_{\alpha=1}^{9} N^{\alpha}(x,y)a^{\alpha}(t)$$

Along boundary $1-4-8$

$x = 0 \Rightarrow$

$N^2 = N^3 = N^5 = N^6 = N^7 = N^9 = 0 \Rightarrow$

$\text{T(0,0.25,t)} = N^1(0,0.25)a^1(t) + N^8(0,0.25)a^8(t) + N^4(0,0.25)a^4(t)$

$$\left.\begin{array}{r} a^8 = 0 \\ a^1 = 0.3 \\ a^4 = 0.72 \\ N^1(0,0.25) = \frac{1}{0.5}(0.25-0.5)(0.25-1) = \frac{3}{8} \\ N^4(0,0.25) = \frac{1}{0.5}0.25(0.25-0.5) = -\frac{1}{8} \end{array}\right\} \Rightarrow T(0,0.25) = 0.3 \cdot \frac{3}{8} - 0.72 \cdot \frac{1}{8} = 0.0225$$

Task 2

Gauss integration is calculated on $(\pm 1, \pm 1, \pm 1)$ which means that we must make a change of variables, i.e.

$$\left.\begin{array}{r} x = 2\xi \\ y = 2\eta \\ z = \frac{1}{2} + \frac{3}{2}\psi \end{array}\right\} \Rightarrow$$

$$\mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \psi} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} & \frac{\partial y}{\partial \psi} \\ \frac{\partial z}{\partial \xi} & \frac{\partial z}{\partial \eta} & \frac{\partial z}{\partial \psi} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \frac{3}{2} \end{bmatrix} \Rightarrow$$

$$\det(\mathbf{J}) = 6$$

$$\mathbf{I} = \int_V f(x,y,z)dV = \int_{V^*} f(\xi,\eta,\psi) \cdot \det(\mathbf{J})dV^* = \int_{V^*} 6f(\xi,\eta,\psi)dV^*$$

Gauss integration, one integration point $(\xi = 0, \eta = 0, \psi = 0)$, weights: $H_\xi = H_\eta = H_\psi = 2$

$$\mathbf{I} = 6f(\xi = 0, \eta = 0, \psi = 0) \cdot H_\xi \cdot H_\eta \cdot H_\psi = 6 \cdot (0 + 0 + \frac{1}{2}) \cdot 8 = 24$$

Task 3 FHL064

See page 326 in (*Finite Element Method. Niels Ottosen, Hans Petersson*)

Task 3 FHLF01

$$\Pi = W - \mathbf{F}^T\mathbf{a}$$

Strain energy W is defined as *energy stored in an elastic body under loading*, which in this case is defined by

$$W(\delta) = \sum_i k_i(u_i)^2 = \frac{1}{2}\mathbf{a}^T\mathbf{K}\mathbf{a} \tag{1}$$

$$\frac{\partial \Pi}{\partial \mathbf{a}} = \mathbf{K}\mathbf{a} - \mathbf{F}$$

$$\mathbf{K} = \begin{bmatrix} k_1 + k_2 & -k_1 & -k_2 & 0 \\ -k_1 & k_1 + k_3 + k_4 & -k_4 & -k_3 \\ -k_2 & -k_4 & k_2 + k_4 + k_5 & -k_5 \\ 0 & -k_3 & -k_5 & k_3 + k_5 \end{bmatrix}$$

$\mathbf{K}$ is positive semidefinite if $\mathbf{a}^T\mathbf{K}\mathbf{a} \geq 0 \ \forall \mathbf{a} \neq 0$ and there exist at least one $\mathbf{a} \neq 0$ such that $\mathbf{a}^T\mathbf{K}\mathbf{a} = 0$. Now looking at (1) we see that $\mathbf{a}^T\mathbf{K}\mathbf{a} \geq 0$ must be fulfilled $\forall \mathbf{a}$ and if we set all the elements in $\mathbf{a}$ equal to the same value (*rigid body movement*) we find that $\mathbf{a}^T\mathbf{K}\mathbf{a} = 0$, i.e we have shown that $\mathbf{K}$ is positive semidefinite.

$$\mathbf{K}\mathbf{a} = \mathbf{F} \Leftrightarrow \begin{bmatrix} k_1 + k_2 & -k_1 & -k_2 & 0 \\ -k_1 & k_1 + k_3 + k_4 & -k_4 & -k_3 \\ -k_2 & -k_4 & k_2 + k_4 + k_5 & -k_5 \\ 0 & -k_3 & -k_5 & k_3 + k_5 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix}$$

Inserting the known data

$$20 \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ u_2 \\ u_3 \\ 0 \end{bmatrix} = \begin{bmatrix} F_1 \\ 0 \\ 10 \\ F_4 \end{bmatrix} \Rightarrow \left\{ \begin{array}{c} 3u_2 - u_3 = 0 \\ -u_2 + 3u_3 = \frac{1}{2} \end{array} \right\} \Rightarrow \begin{array}{c} u_2 = \frac{1}{16} \\ \\ u_3 = \frac{3}{16} \end{array}$$

Task 4

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Task 5

The differential equation:

$$\frac{\partial}{\partial x}\left((1+x^2)\frac{\partial \varphi}{\partial x}\right)+\frac{\partial}{\partial y}\left((1+y^2)\frac{\partial \varphi}{\partial y}\right)+\xi=\dot{\varphi}$$

can be written as:

$$\operatorname{div}\left(\left[\begin{array}{cc} 1+x^2 & 0 \\ 0 & 1+y^2 \end{array}\right]\left[\begin{array}{c} \frac{\partial \varphi}{\partial x} \\ \frac{\partial \varphi}{\partial x} \end{array}\right]\right)+\xi=\dot{\varphi}.$$

Let $\boldsymbol{D}=\left[\begin{array}{cc} 1+x^2 & 0 \\ 0 & 1+y^2 \end{array}\right]$ and $\boldsymbol{q}=\boldsymbol{D}\nabla\varphi$ then we get:

a)

$$\int_A v \operatorname{div}\boldsymbol{q}\, dA+\int_A v\xi\, dA+\int_A v\dot{\varphi}\, dA$$

Using Green-Gauss theorem:

$$\oint_L \boldsymbol{q}^T\boldsymbol{n}\, dL-\int_A(\nabla v)^T\boldsymbol{q}\, dA+\int_A v\xi\, dA=\int_A v\dot{\varphi}\, dA$$

b)
Using: $v=\boldsymbol{Nc}$, $\varphi=\boldsymbol{Na}$, $\dot{\varphi}=\boldsymbol{N}\dot{\boldsymbol{a}}$, $\nabla v=\boldsymbol{Bc}$ and $\nabla\varphi=\boldsymbol{Ba}$, where $\boldsymbol{c}$ is arbitrary, we get:

$$\underbrace{\oint_L \boldsymbol{N}^T q_n\, dL}_{\boldsymbol{f}_b}-\underbrace{\int_A \boldsymbol{B}^T\boldsymbol{D}\boldsymbol{B}\, dA}_{\boldsymbol{K}}\,\boldsymbol{a}+\underbrace{\int_A \boldsymbol{N}^T\xi\, dA}_{\boldsymbol{f}_l}=\underbrace{\int_A \boldsymbol{N}^T\boldsymbol{N}\, dA}_{\boldsymbol{M}}\,\dot{\boldsymbol{a}}$$

c) $T=\alpha_1+\alpha_2 x+\alpha_3 y+\alpha_4 xy$
d) Yes
e) Yes, the $xy-$term.

Task 6

For the triangular element use the approximation:

$$T = \alpha_1 + \alpha_2 x + \alpha_3 y = \begin{bmatrix} 1 & x & y \end{bmatrix} \boldsymbol{\alpha}$$

The C-matrix method gives $\boldsymbol{a} = \boldsymbol{C}\boldsymbol{\alpha}$ which then gives: $T = \begin{bmatrix} 1 & x & y \end{bmatrix} \boldsymbol{C}^{-1}\boldsymbol{a}$, where

$$\boldsymbol{a} = \begin{bmatrix} T_7 \\ T_8 \\ T_{11} \end{bmatrix}, \boldsymbol{C} = \begin{bmatrix} 1 & 7 & 1 \\ 1 & 9 & 1 \\ 1 & 8 & 2 \end{bmatrix} \text{ and } \boldsymbol{\alpha} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$$

The shape functions $\boldsymbol{N} = \begin{bmatrix} 1 & x & y \end{bmatrix} \boldsymbol{C}^{-1}$ Then we get:

$$\boldsymbol{B} = \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \end{bmatrix} \boldsymbol{N} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \boldsymbol{C}^{-1}$$

The stiffness matrix:

$$\begin{aligned} \boldsymbol{K} &= \int_A \boldsymbol{B}^T k \boldsymbol{I} \boldsymbol{B} dA = kA_e \boldsymbol{C}^{-T} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \boldsymbol{C}^{-1} \\ &= kA_e \boldsymbol{C}^{-T} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \boldsymbol{C}^{-1} = \frac{k}{4} \begin{bmatrix} 2 & 0 & -2 \\ 0 & 2 & -2 \\ -2 & -2 & 4 \end{bmatrix} \end{aligned}$$

The load vector:

$$\boldsymbol{f}_l = \int_A \boldsymbol{N}^T Q dA = \boldsymbol{N}(8,\ 1.75) Q_0 = \begin{bmatrix} 1 & 8 & 1.75 \end{bmatrix} \boldsymbol{C}^{-1} Q_0 = \begin{bmatrix} 1 \\ 1 \\ 6 \end{bmatrix} \frac{Q_0}{8}$$