LUNDS UNIVERSITET
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Introduction to the Finite Element Method
Exercises

Mathias Wallin

Division of Solid Mechanics
Lund University, 2013

Report LUTFD2/(TFHF-3086)

1 Introduction

2 Basic linear algebra

Exercise 2.1 The matrix $\boldsymbol{K}$ is defined as

$$\boldsymbol{K} = \alpha \boldsymbol{B}^T \boldsymbol{D} \boldsymbol{B}$$

where α is a scalar and the dimension of $\boldsymbol{B}$ is 3×6.

- Determine the dimension of $\boldsymbol{K}$
- Determine the dimension of $\boldsymbol{D}$
- For the situation where $\boldsymbol{D} = \boldsymbol{D}^T$ show that $\boldsymbol{K}$ is symmetric.

Exercise 2.2 Calculate $det(\boldsymbol{K})$ when $\boldsymbol{K}$ is defined as

$$\boldsymbol{K} = \begin{bmatrix} 1 & 6 & 2 & 3 \\ 0 & 2 & 0 & 0 \\ 1 & 6 & -2 & 1 \\ 0 & 3 & 1 & 2 \end{bmatrix}$$

Exercise 2.3 Consider the quantity $\boldsymbol{a}^T\boldsymbol{K}\boldsymbol{a}$ ($\boldsymbol{K}$ is symmetric) where $dim(\boldsymbol{K})$ is $n \times n$ and $dim(\boldsymbol{a})$ is $n \times 1$. Moreover, $\boldsymbol{a}^T\boldsymbol{K}\boldsymbol{a} \geq 0, \quad \text{and} \quad \boldsymbol{a}^T\boldsymbol{K}\boldsymbol{a} = 0$ for some $\boldsymbol{a} \neq \boldsymbol{0}$

- Determine $det(\boldsymbol{K})$
- Does $\boldsymbol{K}\boldsymbol{x} = \boldsymbol{0}$ have non-trivial solutions ?
- $\boldsymbol{b}$ is known and non zero. How many solutions to $\boldsymbol{K}\boldsymbol{x} = \boldsymbol{b}$ exists ?

Exercise 2.4 In an experiment the variable x is changed and the variable T is measured. The following results are obtained

i	T_i	x_i
1	0.31	0.12
2	0.32	0.15
3	0.34	0.16
4	0.36	0.19

Table 1: Experimental results

A model describing the physical nature of the problem is given by $T = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \alpha_4 x^3$.

- Fit the parameters α_i to the experimental data.

Exercise 2.5 A beam of water with velocity $\boldsymbol{v} = [1\ 3\ 2]^T\ m/s$ is flowing through a surface with area $A = 0.2m^2$ and unit (outward) normal vector $\boldsymbol{n} = \frac{1}{2}\left[\sqrt{3}\ 1\ 0\right]^T$. Calculate the amount of water passing the surface per second.

Exercise 2.6 In a finite element analysis the following linear equation system is obtained

$$\begin{bmatrix} 1 & 6 & -2 & -3 \\ 6 & 2 & -4 & 0 \\ -2 & -4 & 2 & -1 \\ -3 & -2 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ 30 \\ 40 \end{bmatrix}$$

Calculate (by hand) the unkonwns u_3, u_4 and f_1 and f_2.

Exercise 2.7 Use the calfem command solveq to solve Exercise 2.6, i.e.

```
>>x=solveq(K,f,bc)
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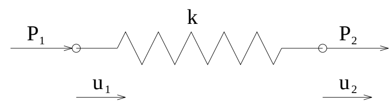
where $\boldsymbol{K}$ is the matrix, $\boldsymbol{f}$ is the right hand side in the system of equations above. Note that $f(1)$ and $f(2)$ are assigned 0 in the function call. Since $x(1) = 2$ and $x(2) = 4$ the variabel $\boldsymbol{bc}$ should be assigned

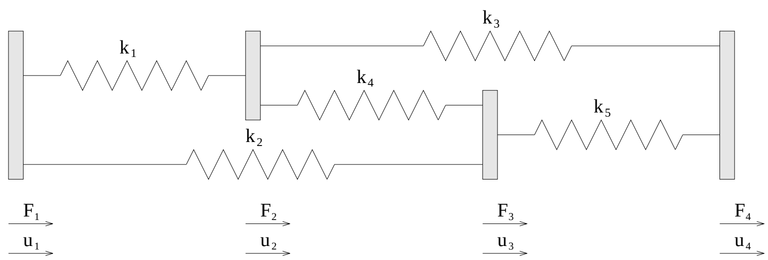
$$\boldsymbol{bc} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

3 Direct approach

Exercise 3.1 a) Derive the stiffness matrix for a spring with the spring stiffness k. The spring is loaded with the forces P_1 and P_2. The nodal displacements are denoted u_1 and u_2.



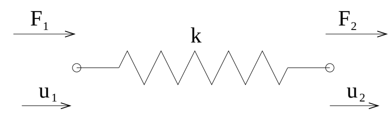
b) Derive the global stiffness matrix, $\mathbf{K}$, for the spring system shown in the figure below



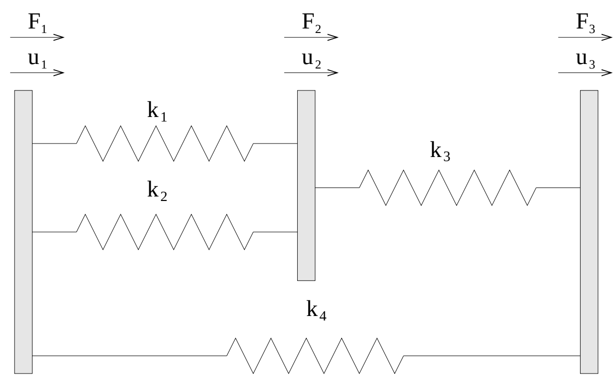
c) Give a physical interpretation of $det(\mathbf{K}) = 0$.

d) For the system given above the following holds:
$u_1 = 1\text{mm}$, $u_4 = 0$, $F_2 = 0\text{N}$, $F_3 = 20\text{N}$
$k_1 = k_2 = k_3 = k_4 = k_5 = 8\text{N/mm}$.
Calculate the displacements u_2 and u_3 and the forces F_1 and F_4 .

Exercise 3.2 a) Derive the element stiffness matrix for a spring with spring constant, k, subjected to the forces F_1 and F_2. The displacements are denoted u_1 and u_2, cf. the figure below.



b) Derive the global load-displacement relation $\boldsymbol{K}\boldsymbol{a} = \boldsymbol{f}$ for the assembly below.



c) For $u_1 = 0$, $F_2 = 0$ och $F_3 = 10\text{N}$ derive F_1, u_2 och u_3 if $k_1 = k_2 = k_3 = k_4 = 10\text{N/mm}$.

d) Find an $\boldsymbol{a} \neq \boldsymbol{0}$ such that $\boldsymbol{a}^T\boldsymbol{K}\boldsymbol{a} = 0$.
Hint: What does $det(\boldsymbol{K}) = 0$ means in terms of boundary conditions.

Exercise 3.3 Consider the assembly of electrical resistances below, cf. Fig.1

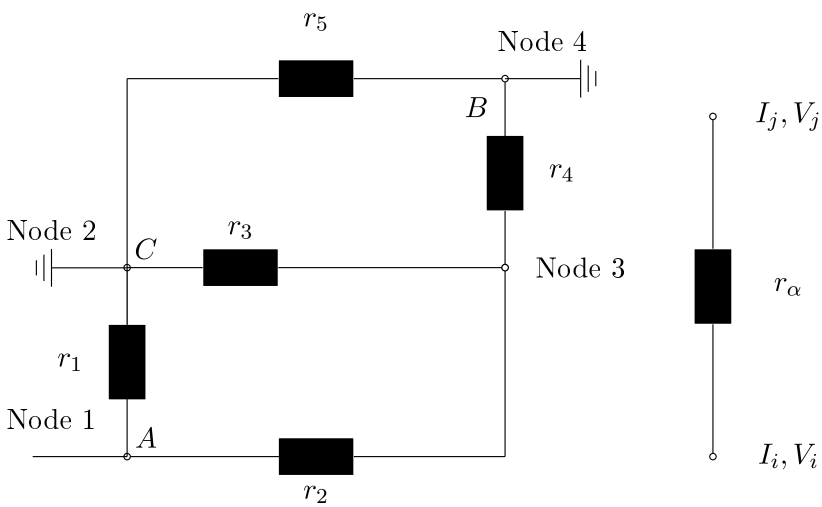


Figure 1: a) Assembly of electrical resistors. b) Typical element of assembly.

If a typical resistance is isolated (cf. Fig.1b), the relation for the current entering the element at the ends, (I_i, I_j) and the end voltages (V_i, V_j) can be written as (Ohm's law)

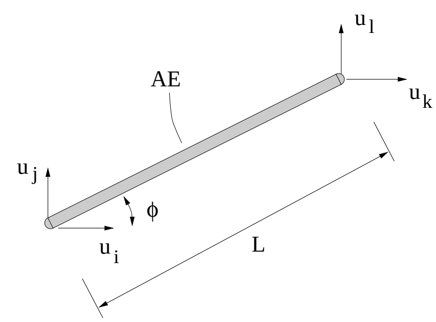
$$\begin{bmatrix} I_i^e \\ I_j^e \end{bmatrix} = \frac{1}{r_\alpha} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} V_i^e \\ V_j^e \end{bmatrix}$$

- Use the "direct method" to establish the system of governing equations, i.e. $\boldsymbol{Ka} = \boldsymbol{f}$, for the situation $r_1 = r_2 = r_3 = r_4 = r_5 = r^*$.
- Solve the system for the situation where the current P=1A is supplied to the system at A. Moreover, the point B and C is connected to ground, $V_B = V_C = 0V$.

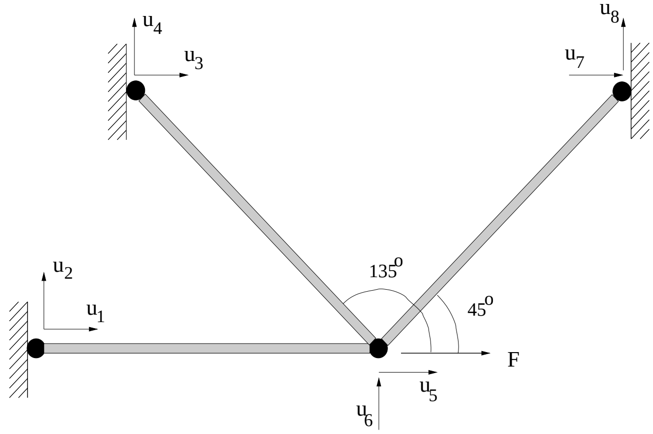
Hint. The Kirchhoff (first) law states that the sum of all currents entering a point is zero.

Exercise 3.4 The element stiffness matrix for a bar is given by

$$\boldsymbol{K}^e = \frac{AE}{L}\begin{bmatrix} \cos^2\phi & \sin\phi\cos\phi & -\cos^2\phi & -\sin\phi\cos\phi \\ \sin\phi\cos\phi & \sin^2\phi & -\sin\phi\cos\phi & -\sin^2\phi \\ -cos^2\phi & -\sin\phi\cos\phi & \cos^2\phi & \sin\phi\cos\phi \\ -\sin\phi\cos\phi & -\sin^2\phi & \sin\phi\cos\phi & \sin^2\phi \end{bmatrix}$$



where the degrees of freedom i, j, k och l corresponds to the rows 1, 2, 3, och 4 in element stiffness matrix. Use this result to establish the global stiffness matrix for the system below



Assume that $k = AE/L$ for the three elements is the same. Morover, note that $sin(45) = sin(135) = cos(45) = -cos(135) = 1/\sqrt{2}$.

4 Strong and weak form- one dimensional heat

Exercise 4.1 Two bodies with different temperatures are located at distance Δx between each other. Give intuitive answers to the questions below:

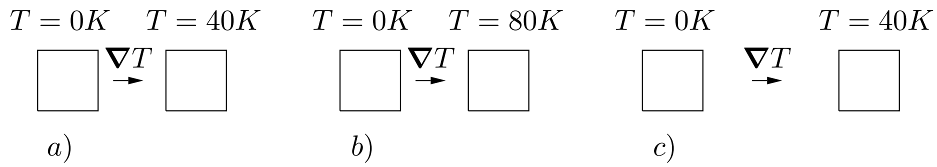
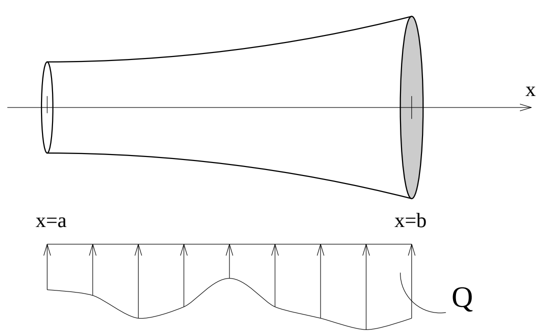


Figure 2: Rectangular discs with given temperature distribution.

- In what direction is the heat flowing in Fig.2 a ?
- The temperature difference between the bodies is doubled. What happens to the temperature derivative ? What happens to the heat flux ?
- The distance between the bodies is doubled. What happens to the temperature derivative ? What happens to the heat flux ?
- Can you conclude your findings ?

Exercise 4.2 Derive the strong form of the (one-dimensional) heat flow problem depicted in the figure below. Derive the weak form corresponding to strong form (leave the boundary



conditions unspecified).

Exercise 4.3 The weak form of the uniaxial heat flow problem is given by

$$\int_0^L \frac{dv}{dx} Ak \frac{dT}{dx} dx = -(vAq)_{x=L} + (vAh)_{x=0} + \int_0^L vQdx, \quad T(x=L) = g$$

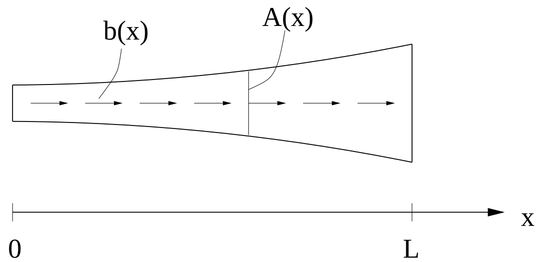
and the strong form of the heat flow problem is given by

$$\frac{d}{dx}\left(Ak\frac{dT}{dx}\right) + Q = 0, \quad 0 \le x \le L$$

$$q(x=0) = -\left(k\frac{dT}{dx}\right)_{x=0} = h, \quad T(x=L) = g$$

Show that the weak form implies that strong form.

Exercise 4.4 A rod is subjected to a distributed force is depicted below.



Show that the strong form of the equilibrium equations can be written as

$$\frac{dN}{dx} + b = 0 \; ; \qquad N = \sigma \, A$$

where N is the normal force and σ the stress.

Derive the weak form of the problem.

Exercise 4.5 An insulated cable with radius R_1 is submerged into the sea. The thickness of the insulation is $R_2 - R_1$, cf. Fig. 3.

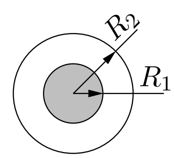


Figure 3: Insulated cable

- Establish the strong form of the one dimensional heat flow equation for the insulation.
- Derive the weak form corresponding to the strong form.

5 Gradient, Gauss divergence theorem and Green-Gauss theorem

Exercise 5.1

- In the three-dimensional situation the spacial variation of a scalar field given by the gradient. What is the 1-D counterpart ?
- In the three-dimensional situation a volume integral of a divergence of a vector field can be transformed into a surface integral (Gauss's theorem). What is the 1-D counterpart ?
- What is the one dimensional counterpart to the Green-Gauss theorem ?

Exercise 5.2 The temperature in a rectangular disc is given by $T(x, y) = ax + by$, cf. Fig. 4.

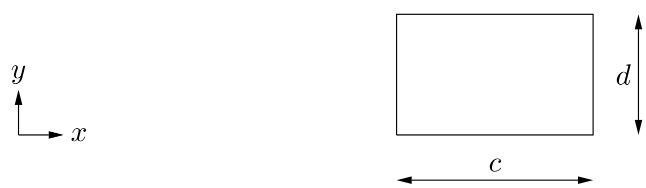


Figure 4: Rectangular disc with given temperature distribution.

a) Calculate the integral $\oint (\boldsymbol{\nabla} T)^T \boldsymbol{n} d\mathcal{L}$ where $\boldsymbol{n}$ is the normal vector to the disc.

b) Calculate the divergence of the temperature gradient.

c) Could the result in a) have been obtained directly from b) ?

Exercise 5.3 For $\phi = x^2 + y + 10$, calculate the line integral

$$\int_{\mathcal{L}} \phi(x, y) d\mathcal{L}$$

where $\mathcal{L}$ is the straight line from $(x, y) = (1, 4)$ to $(5, 1)$. What happens if the integration is performed from $(5, 1)$ to $(1, 4)$ instead ?

Exercise 5.4 A curve in the $x - y$ plane is defined by $\phi = x^2 + y + 10 = 0$. Calculate the normal vector to the curve at $(1, -11)$.

6 Strong and weak form of 2-D and 3-D heat flow

Exercise 6.1 An elliptic disc is defined by

$$\left(\frac{2x-L}{2L}\right)^2 + \left(\frac{2y}{3L}\right)^2 \leq 1$$

The temperature field within the disc is given by

$$T = T_0\left[(x/3L)^2 + (y/L)^2\right]$$

The constitutive law for heat flow is assumed to be given by the law of Fourier, i.e.

$$\boldsymbol{q} = -\boldsymbol{D}\boldsymbol{\nabla}T, \quad \boldsymbol{D} = k\boldsymbol{I}$$

a) Determine the heat flux vector at the boundary point (L/2,3L/2)

b) Determine the normal vector at the boundary point (L/2,3L/2)

c) Determine the heat per unit area leaving the disc at the boundary point (L/2,3L/2)

Exercise 6.2 Let Q be the amount of heat supplied to a body per unit time ($[Q] = [J/m^3 s)$. Let the heat flux per unit time and unit area leaving the body be denoted q_n ($[q_n] = [J/m^2 s]$).

- Derive the global heat balance for the stationary (time-independent) situation.
- Use Gauss divergence theorem to establish the strong form of the heat equation.
- Derive the weak form of the heat flow problem. Assume that q_n is prescribed at a part S_h of the boundary whereas on a part S_g the temperature is prescribed.

Exercise 6.3 For isotropic materials the $\boldsymbol{D}$ matrix in $\boldsymbol{q} = -\boldsymbol{D}\boldsymbol{\nabla}T$ is given by

$$\boldsymbol{D} = \begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}$$

Show that $\boldsymbol{q}$ is parallell to $\boldsymbol{\nabla}T$.

Exercise 6.4 The Fourier's law can be written as

$$\boldsymbol{q} = -\boldsymbol{D}\boldsymbol{\nabla}T$$

where $\boldsymbol{q}$ is the heat flux vector. From experiments we have that the following inequality holds

$$\boldsymbol{q}^T\boldsymbol{\nabla}T < 0 \quad \forall \boldsymbol{\nabla}T \neq \boldsymbol{0}$$

Show that $\boldsymbol{D}^{-1}$ exsits.

Exercise 6.5 Newton convection is given by

$$q_n = \alpha(T - T_{\infty)}$$

What is the mechanical analogy to this boundary condition ?

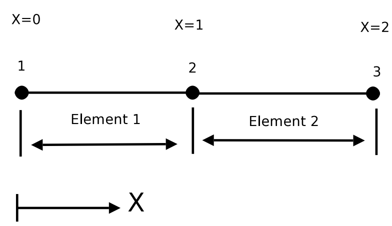
7 Choice of approximating function

Exercise 7.1 For the one dimensional mesh below, determine

a) The element shape functions

b) The global shape functions

Use the C-matrix method as well as Lagrange polynomials. Sketch your result.

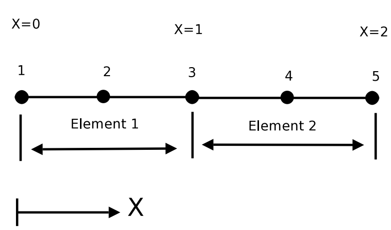


Exercise 7.2 For the one dimensional mesh below, determine

a) The element shape functions

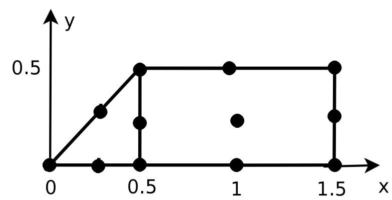
b) The global shape functions

Use the C-matrix method as well as Lagrange polynomials. Sketch your result. The internal node is located at the center of the element.



Exercise 7.3

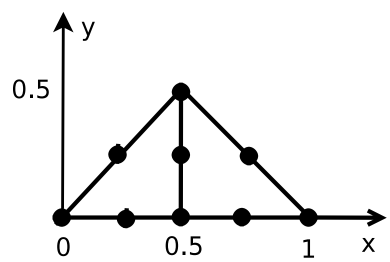
a) An element mesh is based on the following 6-node and 9-node elements



The approximation for the 6-node element is $T = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy + \alpha_5 x^2 + \alpha_6 y^2$, while the approximation for the 9-node element is $T = \beta_1 + \beta_2 x + \beta_3 y + \beta_4 x^2 + \beta_5 y^2 + \beta_6 xy + \beta_7 x^2 y^2 + \beta_8 xy^2 + \beta_9 x^2 y$

Check if the convergence criterion is fulfilled.

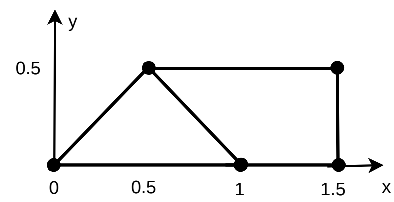
b) An element mesh is based on the following 6 element



The approximation for the 6-node element is $T = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy + \alpha_5 x^2 + \alpha_6 y^2$,

Check if the convergence criterion is fulfilled.

Exercise 7.4 An element mesh is based on the following 3-node and 4-node elements

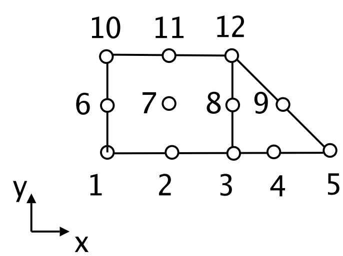


The approximation for the 4-node element is $T = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy$, while the approximation for the 3-node element is $T = \beta_1 + \beta_2 x + \beta_3 y$

a) Is the convergence criterion is fulfilled ?

b) Use the C-matrix method to obtain the element shape functions for the 3-node element.

Exercise 7.5 Due to a chemical reaction, heat is generated within a body. The heat flow is governed by the local balance law $(div(\boldsymbol{q}) - Q = 0)$ where $\boldsymbol{q}$ is the heat flux vector $[J/m^2 s]$ and Q $[J/m^3 s]$is the heat generated by the chemical reaction. The body is modelled by one 9-node element and one 6-node triangular element. Along the boundaries $\mathcal{L}_{1-5}$, $\mathcal{L}_{1-10}$ and $\mathcal{L}_{10-12}$ the body is completely insulated whereas Newton convection applies along $\mathcal{L}_{5-12}$, i.e. $q_n = \alpha(T - T_\infty)$.



The result of the FE-analysis is given below:

$$\boldsymbol{a}^T = [13\ 15\ 18\ 14\ 14\ 16\ 13\ 15\ 18\ 14\ 14\ 19]$$

and the nodal coordinates are given by

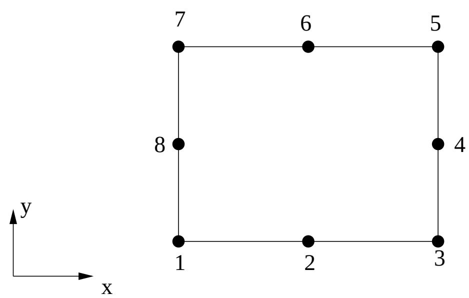
$$\boldsymbol{Coord} = \begin{bmatrix} 0 & b & 2b & 3b & 4b & 0 & b & 2b & 3b & 0 & b & 2b \\ 0 & 0 & 0 & 0 & 0 & b & b & b & b & 2b & 2b & 2b \end{bmatrix}$$

where the first row corresponds to x-coordinates and the second row corresponds to y-coordinates.

Determine the total heat generated within the structure due to the chemical reaction for the situation where the thickness is b and the ambient temperature, $T_\infty = 0$.

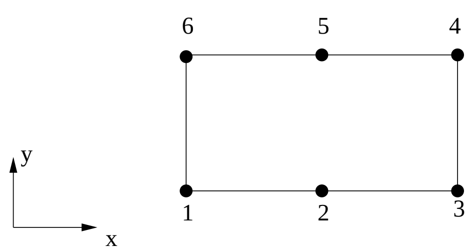
Exercise 7.6 For a two-dimensional thermal analysis, where ϕ is the unknown temperature field consider the following questions:

a) For an 8-node element, ϕ is approximated by a polynomial (element borders are parallel to the coordinate axes).



Suggest a form for ϕ and show that the proposed form fulfills the convergence requirement.

b) Same as for a), but consider now the 6-node element given below.



Exercise 7.7 Consider the four node element below

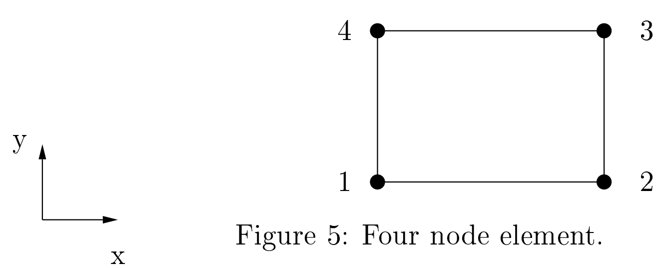


Figure 5: Four node element.

a) Suggest a suitable approximation for a scalar problem for the element above.

b) Does the proposed approximation involve any parasitic terms ?

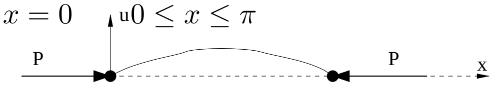
c) Use the **C**-matrix method to obtain the element shape functions. You do not need to calculate the inverse of **C**, i.e. it is sufficient to establish the matrices that are involved in the calculation.

d) What is the value of the element shape function N_2^e at the nodes 1, 2 and 3 ?

8 Choice of weight function

Exercise 8.1 The differential equation governing the response of the column depicted below is given by

$$\frac{d^2u}{dx^2} + \frac{1}{\pi^2}u + \sin x = 0 \qquad 0 \leq x \leq \pi$$
$$u(0) = u(\pi) = 0$$



Adopt the following approximation for the deflection

$$u^{app}(x) = \sum_{k=1}^{n} a_k \sin(kx)$$

1. Derive an expression for the error, $e(x)$
2. Describe the difference between the "point collocation", "subdomain collocation", "least-square" and "Galerkin" methods.
3. For the situation $n = 1$, determine u^{app} using the "point collocation" and "Galerkin" method.

Exercise 8.2 Consider the differential equation

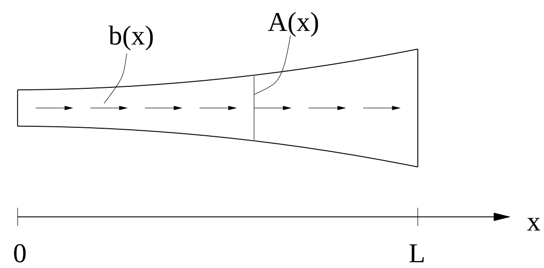
$$\frac{d^2u}{dx^2} + u + 1 = 0$$

together with the boundary conditions $u = 0$ at $x = \pm 1$. An approximative solution should be derived using Galerkin's weighted residual method.

- Suggest a suitable approximation that involves one unknown parameter. The trial function should be chosen as a trigonometric function.
- Determine the unkown parameter using the condition that the weight function is orthogonal to the residual, i.e. $\int_{-1}^{1} e \cdot v dx = 0$

9 FE formulation of one dimensional heat flow

Exercise 9.1 A rod is subjected to a distributed load and a temperature distribution



The equation of equilibrium for the rod can be written as

$$\frac{dN}{dx} + b = 0 \; ; \qquad N = \sigma \, A$$

where N is the normal force and σ the stress.

For the uniaxial case, the constitutive relation for a linear thermoelastic material is given by

$$\sigma = E(\epsilon - \alpha \, \Delta T)$$

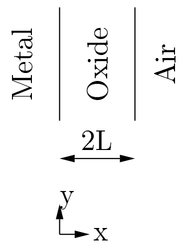
where ΔT is the temperature difference measured for the reference state.

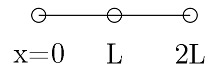
Derive the weak form of the problem and the corresponding FE-formulation according to Galerkin ($\epsilon = du/dx$ where u is the displacement). Then for the following case

$$\begin{array}{ll} u(0) = 0 \; ; & N(L) = 0 \\ b = \text{constant} & \Delta T = T_o(1 + \frac{x}{L}) \end{array}$$

calculate the displacement at $x = L/2$, assuming that E, A, α and T_o are constants. Use two 2-node elements to approximate the rod.

Exercise 9.2 Consider a metal which is covered by a thin film of oxide.





The growth law of the oxide film is governed by Fick's law, i.e.

$$q = -D\frac{dc}{dx}$$

where q is the number of ions diffusing through the cross section per unit area and unit time. The concentration is denoted c. The diffusion parameter, D, is for this oxide film given by

$$D = D_0(1 + 2ax)$$

Moreover, for stationary conditions the ion balance requires

$$-\frac{dq}{dx} + Q = 0$$

where Q is the ion supply per unit volume and unit time.

a) Establish the finite element formulation for the diffusion problem described above. All steps in the derivation should be presented.
b) Using two linear finite elements, calculate the concentration c through the oxide film. The concentration at the oxide/metal surface is c_m whereas the concentration at oxide/air surface is given by c_a. The thickness of the oxide layer is $2L$. D_0 and a are constants in the constitutive law. Moreover no internal supply is present, i.e. $Q = 0$.
c) Modify the boundary conditions at $x = 2L$ such that $q = kc$ and calculate the new stiffness matrix ($\tilde{\boldsymbol{K}}\boldsymbol{a} = \tilde{\boldsymbol{f}}$). Note that the new system does not need to be solved.

Exercise 9.3 A tapered fin of thickness t and length L is exposed to convection along it's length. The temperature distribution for this situation is governed by

$$\frac{d}{dx}\left(x\frac{dT}{dx}\right) - N^2(T - T_\infty) = 0, \quad 0 < x < L \tag{1}$$

where N is a constant (given) parameter defined as $N^2 = \frac{\alpha}{k}\sqrt{1 + \frac{L^2}{Y^2}}$. The ambient (constant) temperature is given by T_∞. Moreover, α represents the convection coefficent and k the conductivity. L and Y are shown in Fig. 6. The boundary conditions are given by

$$\left[x\frac{dT}{dx}\right]_{x=0} = 0, \quad T(L) = T_0$$

a) Determine the weak form of the heat flow problem given by above.

b) Determine the correspondning finite element formulation.

c) For the situation $N^2 = 6/L$, use two linear elements of equal size and calculate $\boldsymbol{K}$ and $\boldsymbol{f}$ in the FE equation $\boldsymbol{K}\boldsymbol{a} = \boldsymbol{f}$.

d) Assuming that $T_\infty = 0^oC$ and $T_0 = 100^oC$, determine the temperature distribution in the fin.

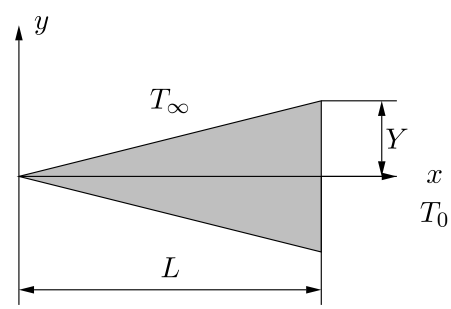


Figure 6: Measures of the tapered fin.

FE formulation of one dimensional transient heat flow - Not covered in the text book

Exercise 9.4 The one dimensional diffusion problem can be written as

$$\frac{d}{dx}\left(k_1\frac{dc}{dx}\right)+Q=k_2\dot{c}, \quad 0\leq x\leq L$$

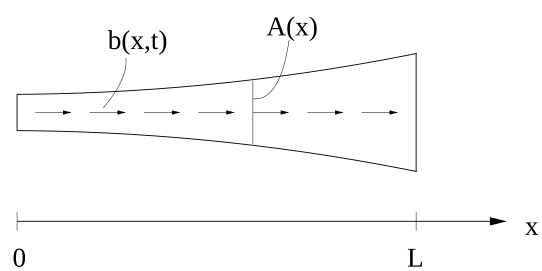
where k_1 and k_2 are constant material parameters and c the concentration. Q represents the (constant) internal generation. The boundary and initial conditions are given by $\frac{dc}{dx}(x=0,t)=q_0$, $c(L,t)=0$ and $c(x,0)=c_0 sin(\pi x/L)$.

a) Derive the FE formulation corresponding to the differential equation above.

b) Use two equally long elements and calculate the element matrices.

c) Describe (in words or equations or both) how the time integration is performed. Describe how the initial condition is introduced.

Exercise 9.5 The governing equation for a rod subjected to a distributed load (per unit length), $b(x,t)$ can be written as



$$\frac{d}{dx}\left(EA\frac{du}{dx}\right)+b=m\ddot{u}, \quad \sigma=E\epsilon=E\frac{du}{dx}$$

where E is the Young's modulus, A cross section area, σ the stress and u the displacement. The mass per unit length is denoted m.

a) Derive the weak form of the problem.

b) Derive finite element formulation.

c) Show that the stiffness matrix is postitive semidefinite if EA is positive.

d) The structure is modeled using three two-node elements of equal length. Calculate the system matrices for the situation $b=b(t)$, $A=const.$, $m=const.$ and $E=const.$.

Note that you do not need to integrate the system over the time.

10 FE formulation of 3D heat flow

Exercise 10.1 In order to analyze the heat exchange from a chimney a FE-analysis is performed. The gas entering the chimney from the stove has the temperature $T = 300^oC$. The chimney as well as a cross section showing the three channels are illustrated in the figures below. Moreover, the FE-mesh that is used in the FE-analysis is also provided. The governing

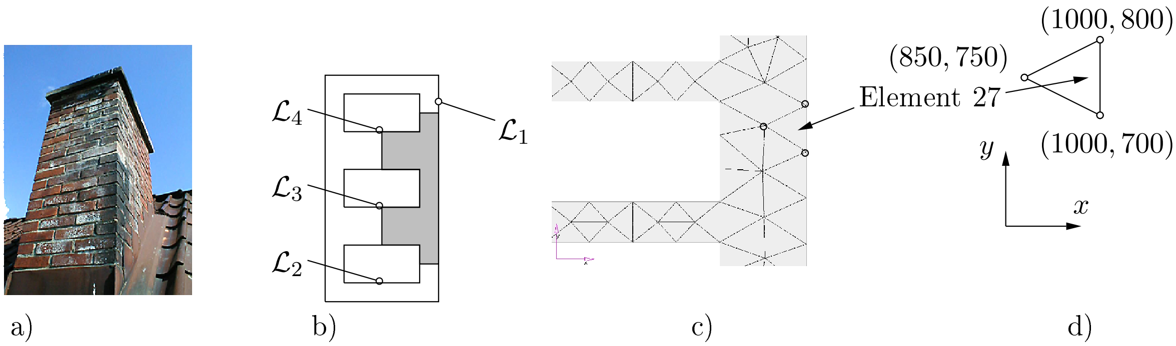


Figure 7: a) Chimney. b) Cross-section of chimney. c) FE-mesh of element 27 and its surrounding elements. d) Location of element 27

equation for the heat flow problem is given by

$$div\boldsymbol{q} = 0, \quad \boldsymbol{q} = -k\boldsymbol{\nabla}T$$

where k represents the constant conductivity, $\boldsymbol{q}$ the heat flux vector and T the temperature.

The boundary condition for the problem is given by

$$\mathcal{L}_1 : q_n = \alpha(T - 22) \quad \mathcal{L}_2 : T = 300^oC \quad \mathcal{L}_3 : q_n = \alpha(T - 22) \quad \mathcal{L}_4 : q_n = \alpha(T - 22)$$

a) Derive the FE-formulation for the two-dimensional heat flow problem.
b) Calculate the element matrices above for element 27, i.e calculate the element stiffness matrix, element force vectors and the matrices that arises from the boundary conditions.

Hint: The following relation holds

$$\begin{bmatrix} 1 & 850 & 750 \\ 1 & 1000 & 700 \\ 1 & 1000 & 800 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{20}{3} & \frac{14}{3} & -\frac{31}{3} \\ -\frac{1}{150} & \frac{1}{300} & \frac{1}{300} \\ 0 & -\frac{1}{100} & \frac{1}{100} \end{bmatrix}$$

Exercise 10.2 Continuity for the current density in a $\mathcal{B}$ requires that

$$div(\boldsymbol{j}) = 0 \quad \text{or} \quad \boldsymbol{\nabla} \cdot \boldsymbol{j} = 0$$

where $\boldsymbol{j}$ represents the current density $\left[A/m^2\right]$. Moreover, the constitutive law (Ohm) relating the current density to the electric field is given by

$$\boldsymbol{j} = \sigma \boldsymbol{E}, \quad \sigma = \sigma(\boldsymbol{x}) > 0$$

Note that σ not necessarily need to be constant througout the body. Moreover, the electric field is obtained from the potential V as $\boldsymbol{E} = -\boldsymbol{\nabla} V$.

- Determine the weak form and the FE formulation ($\boldsymbol{Ka} = \boldsymbol{f}$) for the problem given above.
- Show that $\boldsymbol{K}$ is positive semidefinite as long as no boundary conditions are imposed. (Your statement must be shown.)

Exercise 10.3 In the drying process of timber it is of utmost importance to be able to determine and control the moisture content in the timber. As an example, due to unfavourable moisture distribution, a crack has been formed in the upper of the two boards shown in Fig.8a.

The moisture content is governed by the partial differential equation $div(\boldsymbol{\nabla} m) = 0$ where m is the moisture content measured as $\left[kg\ water/m^3\right]$. A finite element analysis of the board is performed. The board is modeled using 9-node Lagrangian elements. The approximation that is used in the problem is given as

$$m = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2 + \alpha_7 x^2 y + \alpha_8 xy^2 + \alpha_9 x^2 y^2$$

- Does the suggested interpolation involve any parasitic terms ?
- Does the suggested interpolation guarantee convergence ? (Prove your statement !).
- For the element (side length 2) indicated in Fig.8 c) calculate the contribution from the boundary term $\oint_{\mathcal{L}} \boldsymbol{N}^T q_n d\mathcal{L}, \quad q_n = -k(\boldsymbol{\nabla} m)^T \boldsymbol{n}$ (k is a constitutive parameter and $\boldsymbol{n}$ the normal vector to the boundary) to the nodes 1,2 and 3. Moreover, q_n is prescribed to be constant $q_n = q_0$ along the boundary $\mathcal{L}$.

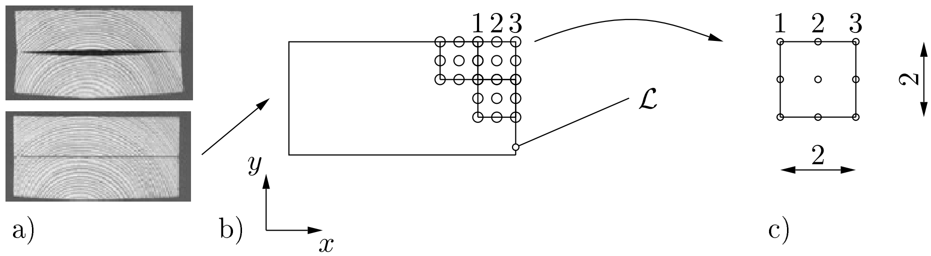
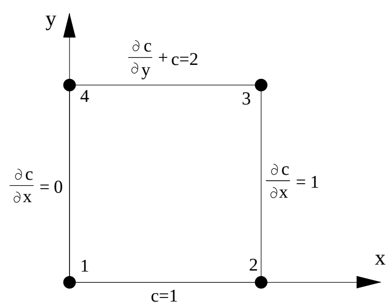


Figure 8: a) Two boards that have been dried under different conditions. b) The boundary and three elements that are used in the finite element analysis of the moisture problem. c) One 9-node Lagrangian element.

Exercise 10.4 For the diffusion problem below, the concentration is given by c ($\left[\text{ions}/m^3\right]$).



In order to simplify the problem it is assumed that the geometry is quadratic with the side length 1. The concentration is governed by the Laplace equation, i.e.

$$\operatorname{div}(\boldsymbol{\nabla} c) = 0$$

supplemented by the boundary conditions

$$\begin{aligned}
&\frac{\partial c}{\partial x} = 0 \quad \text{along} \quad x = 0; \qquad && \frac{\partial c}{\partial x} = 1 \qquad \text{along} \quad x = 1 \\
&c = 1 \qquad \text{along} \quad y = 0; \qquad && \frac{\partial c}{\partial y} + c = 2 \quad \text{along} \quad y = 1
\end{aligned}$$

a) Derive the finite element formulation.
b) Determine the concentration along $y = 1$. The problem shall be solved by using one four node element.

11 Guidelines for element meshes and global nodal nubering

12 Stresses and strains

Exercise 12.1 Derive the strains corresponding the the following displacement fields. Can you describe the displacement fields ? (k_1 and k_2 are constants)

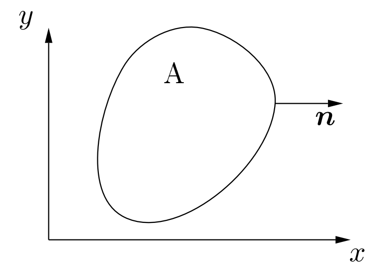
a) $u_x(x,y) = k_1, \quad u_y(x,y) = k_2$

b) $u_x(x,y) = 0, \quad u_y(x,y) = k_1 y$

c) $u_x(x,y) = k_1 y, \quad u_y(x,y) = -k_1 x$

d) $u_x(x,y) = 2k_1 y, \quad u_y(x,y) = 0$

Exercise 12.2 a) Consider a disc (uniform thickness t) subjected to a plane stress state.

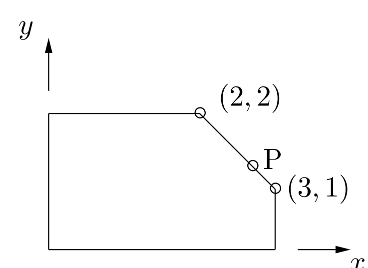


Establish the global equilibrium balance and then derive the local equilibrium equation, i.e.

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + b_x = 0$$
$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + b_y = 0$$

b) For the disc depicted below the stress in P is given by

$$\boldsymbol{S} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} 12 & 6 & 0 \\ 6 & -4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



Determine the traction vector in the point P

c) Determine the normal and shear components of the traction vector in P

13 Linear elasticity

Exercise 13.1 The strain energy is given by $W = \frac{1}{2}\boldsymbol{\sigma}^T\boldsymbol{\epsilon}$. In the situation where plane stress applies show that the out-of-plane strain component ϵ_{zz} does not contribute to the strain energy.

Exercise 13.2 The stiffness tensor $\boldsymbol{D}$ present in the constitutive law $\boldsymbol{\sigma} = \boldsymbol{D}\boldsymbol{\epsilon}$ is given by

$$\boldsymbol{D} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}(1-2\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}(1-2\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(1-2\nu) \end{bmatrix}$$

a) Derive $\boldsymbol{D}$ for plane strain conditions

b) Derive $\boldsymbol{D}$ for plane stress conditions (Note that you need some software for task (e.g. Maple)

14 FE formulation of non-circular shafts

15 Approximating functions for the FE-method-vector problems

Exercise 15.1 For a plane mechanical analysis the displacement field (u_x, u_y) within an element is interpolated as

$$u_x = \sum N_i^e u_{xi}, \quad u_y = \sum N_i^e u_{yi}$$

where u_{xi} and u_{yi} represent the nodal x and y displacements. The small strain components are defined as

$$\epsilon_{xx} = \frac{\partial u_x}{\partial x}, \quad \epsilon_{yy} = \frac{\partial u_y}{\partial y}, \quad \gamma_{xy} = \frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}$$

Establish the $\boldsymbol{N}^e$ and $\boldsymbol{B}^e$ in the matrix relations $\boldsymbol{u} = \boldsymbol{N}^e \boldsymbol{a}^e$ and $\boldsymbol{\epsilon} = \boldsymbol{B}^e \boldsymbol{a}^e$.

16 FE formulation of three dimensional elasticity

Exercise 16.1 For a plane stress analysis the FE-formulation can be written as

$$\int_A \mathbf{B}^T \boldsymbol{\sigma} t dA = \int_A \mathbf{N}^T \mathbf{b} t dA + \int_{\mathcal{L}} \mathbf{N}^T \mathbf{t} t d\mathcal{L}$$

where $\mathbf{b}$ is the body force vector and $\mathbf{t}$ the traction force vector.

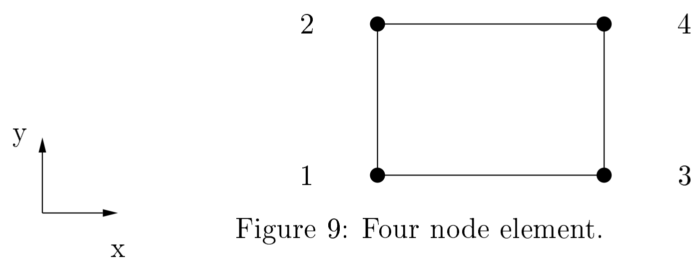


Figure 9: Four node element.

a) Suggest a suitable approximation for the displacement field (u_x, u_y) for the four node element above.

b) Identify, for one element, the size of the matrices (rows x columns) of the matrices $\mathbf{B}$, $\boldsymbol{\sigma}$, $\mathbf{N}$, $\mathbf{b}$ and $\mathbf{t}$ that are present in the finite element formulation.

For a thermoelastic material the Hooke's law can be stated as

$$\boldsymbol{\sigma} = \boldsymbol{D}\left(\boldsymbol{\epsilon} - \boldsymbol{\epsilon}^{\theta}\right), \quad \boldsymbol{\epsilon} = \boldsymbol{B}\boldsymbol{a}$$

c) Derive the matrices in the final FE-formulation $\boldsymbol{K}\boldsymbol{a} = \boldsymbol{f}$ when a plane stress thermo-elastic analysis is performed.

Exercise 16.2 Consider the plane elasticity problem below, cf. Fig.10

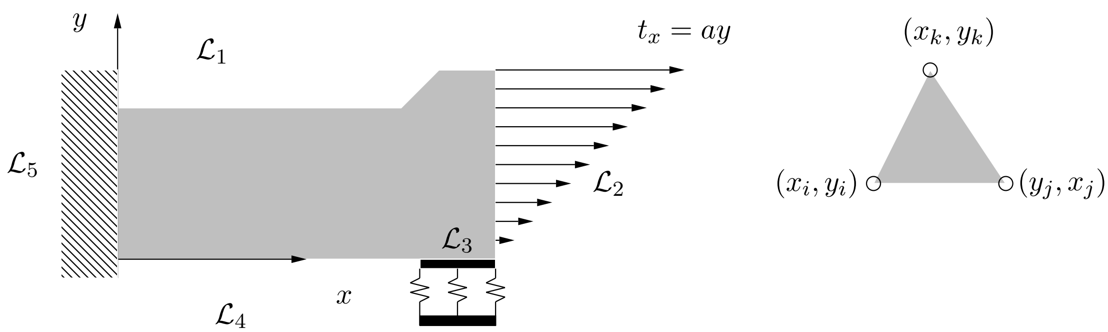


Figure 10: a) Plane elastic problem. b) Typical 3-node element.

The weak form of the equilibrium equations for the structure are given by

$$\int_A (\tilde{\boldsymbol{\nabla}}\boldsymbol{v})^T \boldsymbol{\sigma} t dA = \int_{\mathcal{L}} \boldsymbol{v}^T \boldsymbol{t} t d\mathcal{L} + \int_A \boldsymbol{v}^T \boldsymbol{b} t dA, \qquad \tilde{\boldsymbol{\nabla}} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix}$$

and the constitutive relation is provided by the Hooke's law, i.e.

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{bmatrix} \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \end{bmatrix}, \quad \boldsymbol{\sigma} = \boldsymbol{D}\boldsymbol{\epsilon}, \quad \boldsymbol{D} = \boldsymbol{D}^T$$

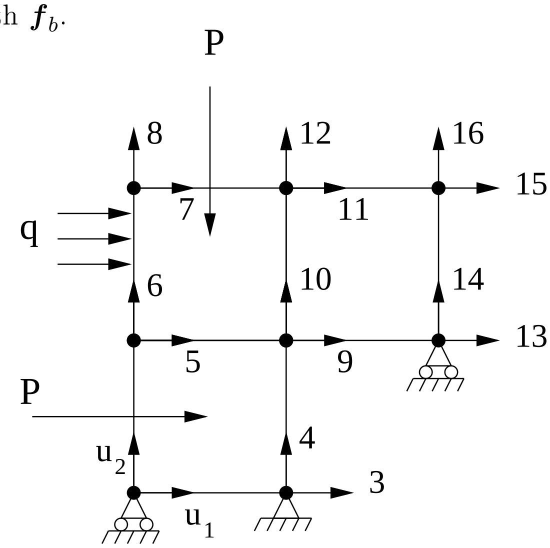
The governing equation for the supporting surface at $\mathcal{L}_3$ is given by

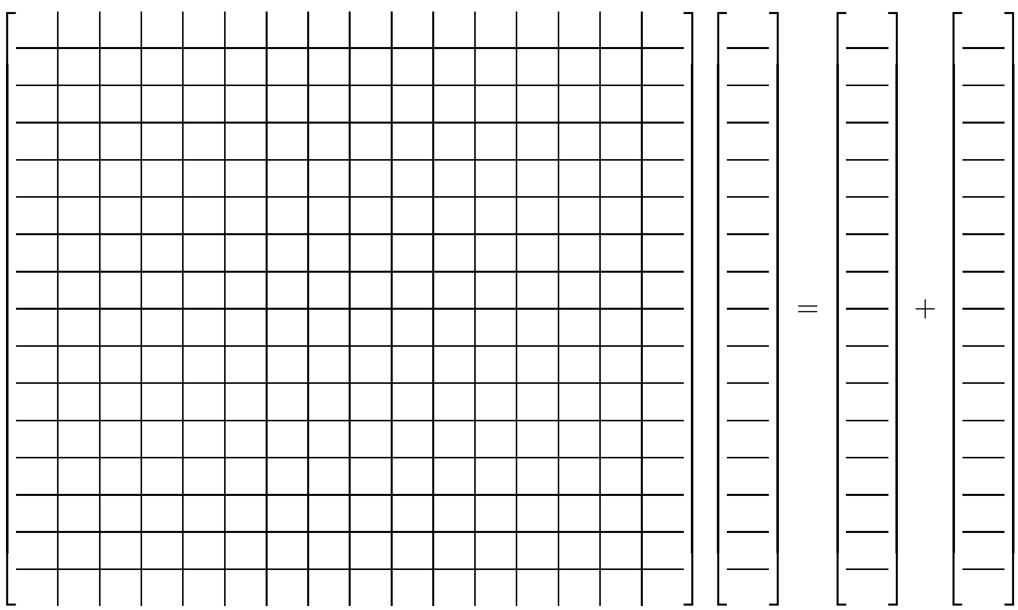
$$t_y = -k u_y,$$

where q denotes the supporting force per square meter and k the spring constant cf. Fig. 10.

- a) Specify the boundary conditions for the structure depicted in Fig.10a.
- b) Derive the FE-formulation for the two-dimensional elasticity problem. Note that the boundary condition at $\mathcal{L}_3$ must be given special attention.
- c) Show that the stiffness matrix, $\boldsymbol{K}$, is symmetric.
- d) For the situation where three node elements are employed, determine the dimension of the matrices $\boldsymbol{K}$, $\boldsymbol{f}$ and $\boldsymbol{a}$ as well as $\boldsymbol{N}^e$, $\boldsymbol{B}^e$, $\boldsymbol{K}^e$, $\boldsymbol{f}^e$. Assume that the finite element mesh consists of $ndof/2$ nodes $ndof$ degrees of freedom and $nelm$ elements.

Exercise 16.3 For a structural problem, a disk is modelled by 4-node elements. The structure has 16 degrees of freedom. The matrix relation found from the FE-formulation is given by $\boldsymbol{Ka} = \boldsymbol{f}_b + \boldsymbol{f}_l$, where the stiffness matrix is denoted $\boldsymbol{K}$, nodal vector $\boldsymbol{a}$, load vector $\boldsymbol{f}_l$ and the boundary vector with $\boldsymbol{f}_b$.



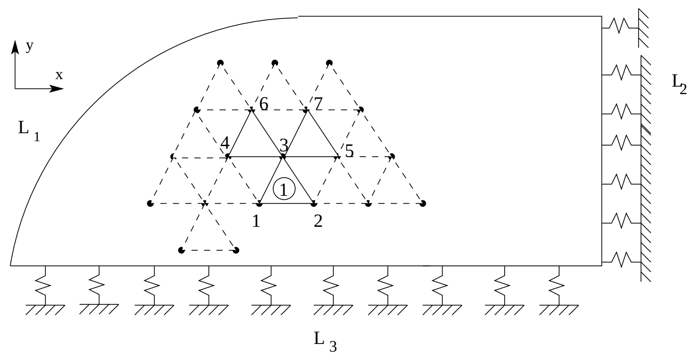


Mark with an x for components known and different from zero, and with 0 for components equal to zero and with ? for unknown components.

Exercise 16.4 A structural analysis of a body subjected to its body weight (g is acting in negative y direction) should be performed using the finite element method. In addition to its body weight it is subjected to an inhomogeneous temperature distribution. The finite element formulation is given by

$$\int_A \boldsymbol{B}^T \boldsymbol{\sigma} dA = \int_{\mathcal{L}} \boldsymbol{N}^T \boldsymbol{t} d\mathcal{L} + \int_A \boldsymbol{N}^T \boldsymbol{b} dA$$

The constitutive law governing the mechanical behaviour is given by the Hooke's law, i.e. $\boldsymbol{\sigma} = \boldsymbol{D}(\boldsymbol{\epsilon} - \boldsymbol{\epsilon}^{\Delta T})$ where $\boldsymbol{\epsilon}^{\Delta T}$ is the thermal strain. The body is supported by two beds of springs. Along L_2, $t_x = -k_x u_x$ where t_x and u_x are components of the traction and displacements vectors. A similar relation along L_3 holds, i.e. $t_y = -k_y u_y$. Note that k_x and k_y are constant parameters.



- Derive the final FE formulation considering the constitutive law and boundary conditions.
- As indicated in the figure three node elements are used in the analysis.

  Show that the three node triangular element satisfies the completeness and the compatibility condition.

17 FE formulation of beams

Exercise 17.1 The equilibrium for a beam is given by

$$\frac{dV}{dx} = -q \quad ; \qquad \frac{dM}{dx} = V$$

where q is the load per unit length, M is the bending moment and V is the shear force.

a) Eliminate the shear force V from the equilibrium equations above and derive an equilibrium equation expressed in terms of the bending moment M.

b) From the equilibrium equation establish the weak form.

c) What are the natural and essential boundary conditions for an arbitrary beam ?

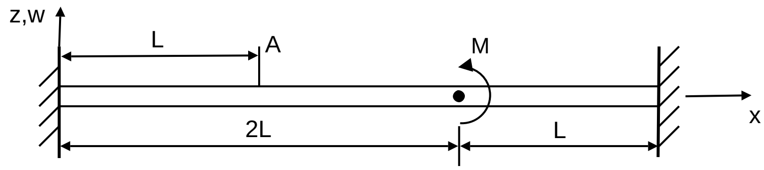
d) The deflection, w, is governed by

$$M = -E^*I^* \frac{d^2w}{dx^2}$$

where $E^*I^* = \frac{1}{A}\int_A Ez^2 dA$ is bending stiffness. Based on b) derive the FE-formulation for an arbitrary beam.

e) Suggest an approximation for w that guarantees that the FE-solution is convergent = compatibility + completeness.

Exercise 17.2 A beam with length $3L$ and bending stiffness EI is rigidly mounted in both ends. A moment M [Nm] is placed at a distance of $2L$ from the left side of the beam, cf. the figure below.



The governing equations for the problem is given by

$$\frac{d^2M}{dx^2} + q = 0 \quad , \qquad M = -EI\frac{d^2w}{dx^2}$$

where q denotes the load intensity (positive in z-direction). Note that the equilibrium equation is derived from the following relations

$$\frac{dM}{dx} = V \quad , \qquad \frac{dV}{dx} = -q$$

a) Derive the weak form of the governing equation, and specify the essential and natural boundary conditions.
b) Derive the FE-formulation for the problem, such that a symmetric stiffness matrix is obtained.
c) Use two elements with the lengths $2L$ and L to determine the deflection at a distance of L form the left side of the beam, i.e. at the point A in the figure. $L = 0.5\ m$, $EI = 7000\ Nm^2$ and $M = 5000\ Nm$. All steps in the calculation should be presented.

Hint:

For a beam with length a the interpolation for the simplest conforming element is given by

$$w = N_1^e u_1 + N_2^e u_2 + N_3^e u_3 + N_4^e u_4$$

where the shape functions are defined as

$$N_1^e = 1 - 3\frac{x^2}{a^2} + 2\frac{x^3}{a^3} \quad , \qquad N_3^e = \frac{x^2}{a^2}(3 - 2\frac{x}{a})$$

$$N_2^e = x(1 - 2\frac{x}{a} + \frac{x^2}{a^2}) \quad , \qquad N_4^e = \frac{x^2}{a}(\frac{x}{a} - 1)$$

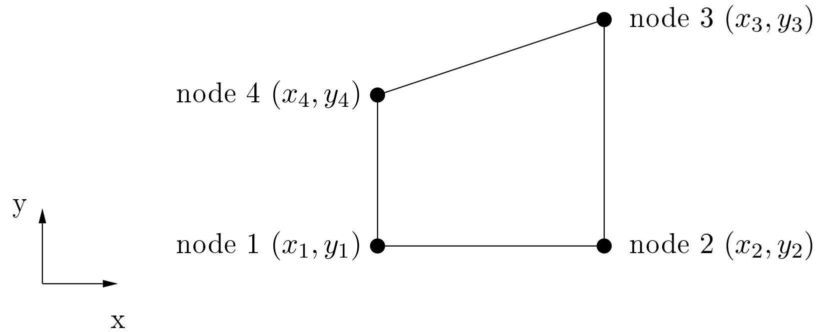
which results in the following element stiffness matrix

$$\boldsymbol{K}^e = \frac{EI}{a^3}\begin{bmatrix} 12 & 6a & -12 & 6a \\ 6a & 4a^2 & -6a & 2a^2 \\ -12 & -6a & 12 & -6a \\ 6a & 2a^2 & -6a & 4a^2 \end{bmatrix}$$

18 FE formulation of plates

19 Isoparametric mapping

Exercise 19.1 A four node element will be used in a thermal analysis according to the figure below.



a) Show that the element above does not satisfy the compatibility condition (no isoparametric mapping is used).

b) Show that the element satisfies the compatibilty conditions if isoparametric mapping is used.
Hint: Isoparametric mapping implies the following mapping of the element coordinates

$$x(\xi, \eta) = \boldsymbol{N}^e(\xi, \eta)\boldsymbol{x}^e, \quad y(\xi, \eta) = \boldsymbol{N}^e(\xi, \eta)\boldsymbol{y}^e$$

where the shape functions are given as

$$N_1^e = \frac{1}{4}(\xi - 1)(\eta - 1), \quad N_2^e = -\frac{1}{4}(\xi + 1)(\eta - 1)$$

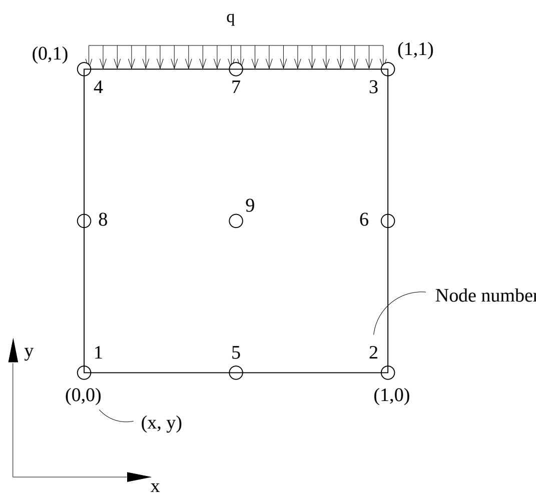
$$N_3^e = \frac{1}{4}(\xi + 1)(\eta + 1), \quad N_4^e = -\frac{1}{4}(\xi - 1)(\eta + 1)$$

Note that the temperature appoximation as usual is given by

$$T = \boldsymbol{N}^e \boldsymbol{a}^e$$

Exercise 19.2 For the problem above, derive the $\boldsymbol{B}$ matrix in terms of the Jacobian and shape functions.

20 Numerical integration



Exercise 20.1 A nine-node Lagrange element will be used for a two-dimensional FE-analysis. The element is subjected to a distributed load, q, cf. the figure above. The boundary load vector is given by $\boldsymbol{f}_b = \int_{\mathcal{L}} t\boldsymbol{N}^T \boldsymbol{t} d\mathcal{L}$, where $\boldsymbol{t}$ is the traction vector and t the thickness. Assume that $t = 1$.

a) Calculate the contribution from q (in the y-direction) to $\boldsymbol{f}_b$ in node 3 and 7 using an analytical integration

b) Calculate the contribution from q (in the y-direction) to $\boldsymbol{f}_b$ in node 3 and 7 using a numerical Gauss integration. The least number of integration points required for an exact integration shall be used.

Exercise 20.2 The stiffness matrix for a four node isoparametric element for finite element analysis is given by

$$\boldsymbol{K}^e = \int_{\eta=-1}^{1} \int_{\xi=-1}^{1} \left[\frac{\partial \boldsymbol{N}^{eT}}{\partial \xi} \frac{\partial \boldsymbol{N}^{eT}}{\partial \eta} \right] \boldsymbol{J}^{-1} \boldsymbol{D} (\boldsymbol{J}^{-1})^T \begin{bmatrix} \dfrac{\partial \boldsymbol{N}^e}{\partial \xi} \\ \dfrac{\partial \boldsymbol{N}^e}{\partial \eta} \end{bmatrix} det(\boldsymbol{J}) d\xi d\eta$$

Assume that $\boldsymbol{D} = \boldsymbol{I}$ and that the nodal coordinates are given by

$$\boldsymbol{x}^T = [10\ 14\ 14\ 10] \qquad \boldsymbol{y}^T = [10\ 10\ 14\ 14]$$

- Compute the component K_{11} using numerical integration (2x2 Gauss points).

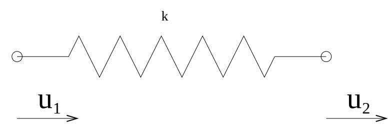
Exercise 20.3 The function f is defined as

$$f(x) = x + 1 + 3x^2 - 2x^3$$

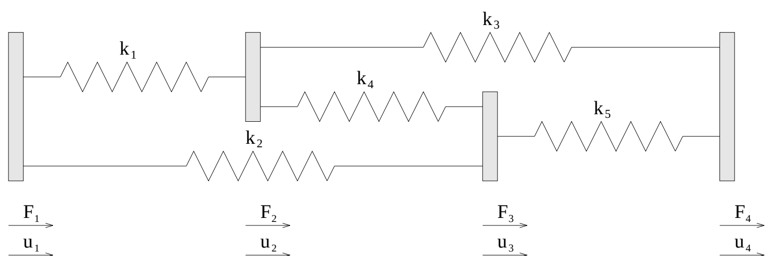
- Use Gauss quadrature (two integration points) to evaluate $I = \int_0^1 f(x)dx$.
- Is the result exact ?
- Comment upon the result.

21 Variational principles

Exercise 21.1 a) Derive the strain energy for a spring with the spring stiffness k. The nodal displacements are denoted u_1 and u_2.



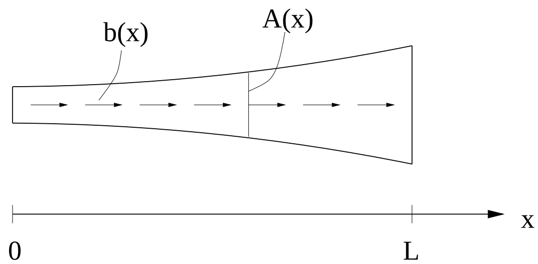
b) For the situation $u_1 = 0$, derive the total strain energy for the spring system shown in the figure below



c) Establish the potential, Π to the system i.e., Strain energy minus potential due to external forces.

d) Show that minimization of the potential Π yields the equlibrium equations.

Exercise 21.2 A rod is subjected to a distributed as depicted below.



The boundary conditions for the problem is given by

$$N(L) = F \quad \text{and} \quad u(0) = 0$$

The principle of virtual work for the rod (for $b = 0$) can be formulated as

$$\int_0^L \sigma A \delta\epsilon dx - \delta u(L) F = 0, \quad \forall \delta u(0) = 0$$

where $\delta\epsilon = \frac{d\delta u}{dx}$.

Show that the principle of virtual work implies the boundary condition and the equilibrium equation $\frac{d(\sigma A)}{dx} = 0$

Solutions:

2 Basic linear algebra

Solution 2.1

a) $Dim(\boldsymbol{K}) = [6x6]$

b) $Dim(\boldsymbol{D}) = [3x3]$

c) $\boldsymbol{K}^T = (\alpha \boldsymbol{B}^T \boldsymbol{D} \boldsymbol{B})^T = \alpha (\boldsymbol{B})^T (\boldsymbol{D})^T (\boldsymbol{B}^T)^T = \alpha \boldsymbol{B}^T \boldsymbol{D} \boldsymbol{B} = \boldsymbol{K}$

Solution 2.2 Expand along 2:nd row: $det(\boldsymbol{K}) = 2(-1)^{2+2} \begin{vmatrix} 1 & 2 & 3 \\ 1 & -2 & 1 \\ 0 & 1 & 2 \end{vmatrix} = -12$

Solution 2.3

a) $\boldsymbol{K}$ is positive semi-definite, i.e. $det(\boldsymbol{K}) = 0$.

b) Non-trivial solutions exists.

c) No or infinite many solutions.

Solution 2.4

$$\begin{array}{l} T_1 = \alpha_1 + \alpha_2 x_1 + \alpha_3 x_1^2 + \alpha_4 x_1^3 \\ T_2 = \alpha_1 + \alpha_2 x_2 + \alpha_3 x_2^2 + \alpha_4 x_2^3 \\ T_3 = \alpha_1 + \alpha_2 x_3 + \alpha_3 x_3^2 + \alpha_4 x_3^3 \\ T_4 = \alpha_1 + \alpha_2 x_4 + \alpha_3 x_4^2 + \alpha_4 x_4^3 \end{array}$$

Arrange on matrix format and solve for $\alpha_1 - \alpha_4$, i.e. $\boldsymbol{\alpha} = \boldsymbol{A}^{-1} \boldsymbol{T} = \begin{bmatrix} \alpha_1 = 0.0041 \\ \alpha_2 = -0.0765 \\ \alpha_3 = 0.5024 \\ \alpha_4 = -1.0714 \end{bmatrix} \cdot 10^3$

Solution 2.5 Project the velocity vector on the normal to the surface,

$$q = A \boldsymbol{v}^T \boldsymbol{n} = 0.4732 m^3/s$$

Solution 2.6 Partition of the system yields

$$\begin{bmatrix} \boldsymbol{A}_1 & \boldsymbol{A}_2 \\ \boldsymbol{A}_3 & \boldsymbol{A}_4 \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{u} \end{bmatrix} = \begin{bmatrix} \boldsymbol{f} \\ \boldsymbol{y} \end{bmatrix}$$

which can be solved to yield $\boldsymbol{u} = [51.3 \ 52.66]^T$ and $\boldsymbol{f} = [-235 \ \ -185]^T$

3 Direct approach

Solution 3.1 a) The force in the spring, N, can be expressed as $N = k(u_2 - u_1)$ and

$$P_1 = -N, \qquad P_2 = N \quad \text{which can be written as} \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}$$

b) The global system of equation is given by

$$\begin{bmatrix} k_1+k_2 & -k_1 & -k_2 & 0 \\ -k_1 & k_1+k_3+k_4 & -k_4 & -k_3 \\ -k_2 & -k_4 & k_2+k_4+k_5 & -k_5 \\ 0 & -k_3 & -k_5 & k_5+k_3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix}$$

With the numerical values inserted the system of equations that shall be solved is formed as

$$8 \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ u_2 \\ u_3 \\ 0 \end{bmatrix} = \begin{bmatrix} F_1 \\ 0 \\ 20 \\ F_4 \end{bmatrix}$$

The solution is given by

$$u_2 = 0.81\ mm, \qquad u_3 = 1.44\ mm, \qquad F_1 = -2\ N, \qquad F_4 = -18\ N$$

Solution 3.2

a)

$$\begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

b)

$$\boldsymbol{K} = \begin{bmatrix} k_1+k_2+k_4 & -k_1-k_2 & -k_4 \\ -k_1-k_2 & k_1+k_2+k_3 & -k_3 \\ -k_4 & -k_3 & k_3+k_4 \end{bmatrix}, \quad \boldsymbol{a} = \begin{bmatrix} 0 \\ u_2 \\ u_3 \end{bmatrix}, \quad \boldsymbol{f} = \begin{bmatrix} F_1 \\ 0 \\ 10 \end{bmatrix}$$

c)

$$\begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

d) $\boldsymbol{a} = [1\ 1\ 1]^T$ results in $\boldsymbol{a}^T\boldsymbol{K}\boldsymbol{a} = 0$. Rigid body motion is not prevented.

Solution 3.3 Direct assembly results in the system

$$\frac{1}{r^*}\begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}\begin{bmatrix} V_1 \\ 0 \\ V_3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ I_2 \\ 0 \\ I_4 \end{bmatrix}$$

which gives $V_1 = 0.6r^* \quad V_3 = 0.2r^*$

Solution 3.4

$$\boldsymbol{K}^e_{1256} = k\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \boldsymbol{K}^e_{3456} = \frac{k}{2}\begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}, \boldsymbol{K}^e_{5678} = \frac{k}{2}\begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix}$$

The total stiffness matrix is given by

$$\boldsymbol{K} = \frac{k}{2}\begin{bmatrix} 2 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 & -1 & 0 & 0 \\ -2 & 0 & -1 & 1 & 4 & 0 & -1 & -1 \\ 0 & 0 & 1 & -1 & 0 & 2 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 & -1 & 1 & 1 \end{bmatrix}$$

4 Strong and weak form- one dimensional heat

Solution 4.1

- Heat is flowing from the 'warm' region to the 'cool' region.
- Doubled temperature difference implies a doubled heat flow.
- Doubled distance reduce the heat flow by a factor of two.
- To summarize we have that the heat flow, q is given by $q \propto \Delta T \frac{1}{\Delta x}$. Compare to Fourier's law $q = -k\frac{dT}{dx}$.

Solution 4.2 Strong form: See course book, pp. 49-51. Weak form: See course book, pp. 56-57.

Solution 4.3 Consult the course book, pages 57-59.

Solution 4.4 Weak form:

$$\int_0^L \frac{dv}{dx} N dx - [vN]_0^L - \int_0^L vb dx = 0$$

Solution 4.5 Balance of a ring of the insulation assuming stationary conditions results in $\frac{d}{dr}(qr) = 0$. Multiply by arbitrary weight function and integrate over the body

$$[vqr]_{R_1}^{R_2} - \int_{R_1}^{R_2} \frac{dv}{dr} rq dr = 0$$

5 Gradient, Gauss divergence theorem and Green-Gauss theorem

Solution 5.1

- The gradient of a scalar field degenerates to a derivative in a one dimensional case.
- The gauss theorem states that $\int_V div(\boldsymbol{q})dV = \int_{\mathcal{S}} q_n d\mathcal{S}$. The one-dimensional counterpart is simply $\int_a^b \frac{df}{dx}dx = f(b) - f(a)$ i.e. "derivative, divergence" is transformed into a boundary term "function value in end-points, line integral".
- In the multi-dimensional situation an integral $\int_A \varphi div(\boldsymbol{q})dV$ can be transformed into $\int_{\mathcal{S}} \varphi \boldsymbol{q}^T \boldsymbol{n} dS - \int_V (\boldsymbol{\nabla}\varphi)^T \boldsymbol{q} dV$ which reduces to the 'integration by parts' in the one dimensional case, i.e. $\int_a^b \frac{df}{dx} g dx = [fg]_a^b - \int_a^b f \frac{dg}{dx} dx$

Solution 5.2

a) The gradient is given by $\boldsymbol{\nabla}T = [a \quad b]^T$. Integration along the boundary results in $\oint_{\mathcal{L}} (\boldsymbol{\nabla}T)^T \boldsymbol{n} d\mathcal{L} = 0$

b) $div(\boldsymbol{\nabla}T) = 0$

c) $\int_{\mathcal{L}} (\boldsymbol{\nabla}T)^T \boldsymbol{n} d\mathcal{L} = \int_A div(\boldsymbol{\nabla}T) dA = 0$

Solution 5.3 Line is parametrized as $y = 4 - (3x - 3)/4$

$$\int_{\mathcal{L}} (x^2 + y(x) + 10) \sqrt{1 + \left(\frac{dy}{dx}\right)^2} |dx| = \sqrt{1 + \left(\frac{3}{4}\right)^2} \int_{\mathcal{L}} (x^2 + y(x) + 10)|dx| = \frac{685}{6}$$

Solution 5.4 The normal is given by $\boldsymbol{n} = \frac{\boldsymbol{\nabla}\phi}{|\boldsymbol{\nabla}\phi|}$, i.e. $\boldsymbol{n} = \frac{1}{\sqrt{5}} [2 \ 1]^T$

6 Strong and weak form of 2-D and 3-D heat flow

Solution 6.1

a) The gradient of the temperature field is given by $\boldsymbol{\nabla} T = \frac{T_0}{L^2}[\frac{2x}{9} \;\; 2y]^T$. The heat flux vector is given by $\boldsymbol{q} = -\frac{kT_0}{L^2}[\frac{2x}{9} \;\; 2y]^T$ which at $(L/2, 3L/2)$ becomes $\boldsymbol{q} = -\frac{kT_0}{L}[\frac{1}{9} \;\; 3]^T$

b) The normal to the surface is given by $\boldsymbol{n} = \boldsymbol{\nabla} g/||\boldsymbol{\nabla} g||$ where $g(x,y) = \left(\frac{2x-L}{2L}\right)^2 + \left(\frac{2y}{3L}\right)^2$. Evaluation yields $\boldsymbol{n} = [0 \;\; 1]^T$

c) The heat flux is given by $q_n = \boldsymbol{q}^T\boldsymbol{n} = -\frac{3kT_0}{L}$

Solution 6.2

a) Heat generated within the body=Heat leaving the body. This balance principle can be formulated as

$$\int_V Q dV = \int_S q_n dS$$

where V and S represents the volume and boundary to the body.

b) Using that $q_n = \boldsymbol{q}^T\boldsymbol{n}$

$$\int_V Q dV = \int_S q_n dS = \int_S \boldsymbol{q}^T\boldsymbol{n} dS = \int_V div(\boldsymbol{q}) dV$$

Since the volume V can be chosen arbitrarily we obtain the local form as

$$div(\boldsymbol{q}) - Q = 0$$

which is the strong form to the problem.

c) Multiply by an arbitrary weight function and integrate over the entire body. Use of the Green-Gauss's theorem results in the weak form. Consult the course book page 85.

Solution 6.3 Since the Fourier's law states that $\boldsymbol{q} = -\boldsymbol{D}\boldsymbol{\nabla} T = -k\boldsymbol{\nabla} T$ it follows that $\boldsymbol{q}$ is parallell to $\boldsymbol{\nabla} T$.

Solution 6.4 The inequality shows that $\boldsymbol{D}$ is positive definite. This condition implies that $\boldsymbol{D}^{-1}$ exsists, cf. course book page 23.

Solution 6.5 A spring is the mechanical analogy (Force is proportional to extension the, i.e. $F = k(u - u_0)$

7 Choice of approximating function

Solution 7.1 Element shape functions:

$$N_1^1 = 1 - x, \quad N_2^1 = x$$

$$N_1^2 = 2 - x, \quad N_2^2 = -1 + x$$

Global shape functions:

$$N_1 = \begin{cases} 1 - x, & 0 \le x \le 1 \\ 0, & 1 < x \le 2 \end{cases}$$

$$N_2 = \begin{cases} x, & 0 \le x \le 1 \\ 2 - x, & 1 < x \le 2 \end{cases}$$

$$N_3 = \begin{cases} 0, & 0 \le x \le 1 \\ -1 + x, & 1 < x \le 2 \end{cases}$$

Solution 7.2 Element shape functions (element 1)

$$N_1^1 = 2(1 - x)(0.5 - x), \quad N_2^1 = 4x(1 - x), \quad N_3^1 = -2x(0.5 - x)$$

Element shape functions (element 2)

$$N_1^2 = 2(1.5 - x)(2 - x), \quad N_2^2 = 4(x - 1)(2 - x), \quad N_3^2 = -2(x - 1)(1.5 - x)$$

Global shape functions

$$N_1 = \begin{cases} N_1^1, & 0 \le x \le 1 \\ 0, & 1 < x \le 2 \end{cases}$$

$$N_2 = \begin{cases} N_2^1, & 0 \le x \le 1 \\ 0, & 1 < x \le 2 \end{cases}$$

$$N_3 = \begin{cases} N_3^1, & 0 \le x \le 1 \\ N_1^2, & 1 < x \le 2 \end{cases}$$

$$N_4 = \begin{cases} 0, & 0 \le x \le 1 \\ N_2^2, & 1 < x \le 2 \end{cases}$$

$$N_5 = \begin{cases} 0, & 0 \le x \le 1 \\ N_3^2, & 1 < x \le 2 \end{cases}$$

Solution 7.3

a) Convergence guaranteed.

b) Convergence guaranteed.

Solution 7.4 a) The completeness requirement is fulfilled for both elements. To fulfill the compatibility requirement the approximated field must be continuous, i.e. the approximation must be uniquely determined by the nodal values on the boundaries. This is not satisfied for the current configuration, i.e. compatibility is not satisfied.

b) The C-matrix method

$$\boldsymbol{N} = \bar{\boldsymbol{N}}\boldsymbol{C}^{-1} = [1 \quad x \quad y]\boldsymbol{C}^{-1}; \quad \boldsymbol{C} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0.5 & 0.5 \end{bmatrix}$$

which results in the following shape functions

$$\boldsymbol{N} = [1 - x - y \quad x - y \quad 2y]$$

Solution 7.5 Gauss's theorem yields $\int_{\mathcal{S}} q_n dS = \int_V div(\boldsymbol{q}) dV = \int_V Q dV =$ Total heat generated within the body. The only contribution to $\int_S q_n dS = \int_{\mathcal{L}} q_n t d\mathcal{L}$ is from $\mathcal{L}_{5-12}$. Introducing an axis (η) along $5 - 12$ starting in 12 allow us to write the temperature distribution as

$$T(\eta) = T_{12}(\eta - \sqrt{2}b)(\eta - 2\sqrt{2}b)/(4b^2) - T_9\eta(\eta - 2\sqrt{2}b)/(2b^2) + T_5\eta(\eta - \sqrt{2}b)/(4b^2)$$

Total heat is given as $\int_0^{2\sqrt{2}b} \alpha T(\eta) b d\eta = \alpha\frac{\sqrt{2}b^2}{3}(T_{12} + 4T_9 + T_5) = \alpha\frac{\sqrt{2}b^2}{3}(19 + 4 \cdot 18 + 14) = 35\alpha\sqrt{2}b^2$

Solution 7.6

a) $\phi = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy + \alpha_5 x^2 + \alpha_6 y^2 + \alpha_7 x^2 y + \alpha_8 xy^2$. For $x = const$ we obtain $\phi = \beta_1 + \beta_2 y + \beta_3 y^2$, i.e. three parameters and three nodes, i.e. a unique temperature distribution is obtained. Similar arguments holds for the y direction.

b) $\phi = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy + \alpha_5 x^2 + \alpha_6 x^2 y$. For $y = const$ we obtain $\phi = \beta_1 + \beta_2 x + \beta_3 x^2$, i.e. three parameters and three nodes, i.e. a unique temperature distribution is obtained.

   For $x = const$ we obtain $\phi = \beta_1 + \beta_2 y$, i.e. two parameters and two nodes, i.e. a unique temperature distribution is obtained.

Solution 7.7

a)

$$\phi = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy$$

b)

$$\alpha_4 xy$$

c)

$$\boldsymbol{N} = \bar{\boldsymbol{N}}\boldsymbol{C}^{-1}$$

where

$$\bar{\boldsymbol{N}} = \begin{bmatrix} 1 & x & y & xy \end{bmatrix} \quad \text{and} \quad \boldsymbol{C} = \begin{bmatrix} 1 & x_1 & y_1 & x_1 y_1 \\ 1 & x_2 & y_2 & x_2 y_2 \\ 1 & x_3 & y_3 & x_3 y_3 \\ 1 & x_4 & y_4 & x_4 y_4 \end{bmatrix}$$

d)

$$N_2^e(x_1, y_1) = 0, \quad N_2^e(x_2, y_2) = 1, \quad N_2^e(x_3, y_3) = 0$$

8 Choice of weight function

Solution 8.1

a)

$$\frac{d^2u}{dx^2} + \frac{1}{\pi^2}u + sin(x) = 0, \quad \Rightarrow L(u) + g(x) = 0$$

where

$$L = \frac{d^2}{dx^2} + \frac{1}{\pi^2}, \quad g(x) = sin(x)$$

The residual is given by

$$e = L(u^{app}) + g(x)$$

b)
- Point collocation: Enforce $e = 0$ at specific points
- Subdomain collocation: Enforce $\int ve\ dx = 0$ over a specific region
- Least square: Minimize the integral $\int e^2\ dx$
- Galerkin: weight function= trial function

c) Specific approximation

$$\boldsymbol{\psi} = \{sin(x)\}, \quad \boldsymbol{a} = \{a\} \quad \Rightarrow \quad L(\boldsymbol{\psi}) = -sin(x) + \frac{sin(x)}{\pi^2}$$

Point collocation, $V = \delta(x - \pi/2)$ results in

$$\left(-sin(\pi/2) + \frac{sin(\pi/2)}{\pi^2}\right) a = -sin(\pi/2) \qquad \Rightarrow \qquad a = \frac{-\pi^2}{1-\pi^2}$$

Galerkin, $V = sin(x)$ results in

$$\int_0^\pi -sin^2(x) + \frac{sin^2(x)}{\pi^2}\ dx\ a = -\int_0^\pi sin^2(x) \quad \Rightarrow \quad a = \frac{-\pi^2}{1-\pi^2}$$

Note: Correct solution $u = \frac{-\pi^2}{1-\pi^2} sin(x)$

Solution 8.2

- The approximation must satisfy the boundary conditions. Choose $u^{app} = a\cos(\frac{\pi}{2}x)$
- Error is defined as $e = \frac{d^2u^{app}}{dx^2} + u^{app} + 1$.
  Insertion into the orthogonal condition yields $a = \frac{-4}{\pi(1-(\frac{\pi}{2})^2)}$

9 FE formulation of one dimensional heat flow

Solution 9.1 Multiply the balance equation by an arbitrary weight function, v and integrate over the entire body. The result is:

$$[vN]_0^L - \int_0^L \frac{dv}{dx} N dx + \int_0^L vbdx = 0$$

which is the weak form of the problem. Using the approximation $u = \boldsymbol{Na}$ as well as the Galerkin choice of weight function $v = \boldsymbol{Nc}$ results in

$$\boldsymbol{c}^T \left[[\boldsymbol{N}^T N]_0^L - \int_0^L \boldsymbol{B}^T N dx + \int_0^L \boldsymbol{N}^T bdx \right] = 0$$

where is was used that $\boldsymbol{\epsilon} = \frac{du}{dx} = \frac{d}{dx}\boldsymbol{Na} = \boldsymbol{Ba}$. Using that $\boldsymbol{c}$ is arbitrary along with $N = AE(\epsilon - \alpha\Delta T) = AE(\boldsymbol{Ba} - \alpha\Delta T)$ results in finite element formulation

$$\int_0^L \boldsymbol{B}^T AE\boldsymbol{B} dx\boldsymbol{a} = [\boldsymbol{N}^T N]_0^L + \int_0^L \boldsymbol{N}^T bdx + \int_0^L \boldsymbol{B}^T AE\alpha\Delta T dx$$

or

$$\boldsymbol{Ka} = \boldsymbol{f}_b + \boldsymbol{f}_l + \boldsymbol{f}_0$$

Using two linear elements results in:

Element 1: $N_1^e = 1 - \frac{2x}{L}$, $N_2^e = \frac{2x}{L}$, $\boldsymbol{B}^e = \frac{2}{L}[-1 \ 1]$ and $\boldsymbol{K}_1^e = \frac{2AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \boldsymbol{K}_2^e$

which gives the total stiffness matrix as $\boldsymbol{K} = \frac{2AE}{L}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$

Load vector: $b = const$ results in $\boldsymbol{f}_l = \frac{bL}{4}[1 \ 2 \ 1]^T$
Load vector due to thermal strains, $\boldsymbol{f}_0$:

$$\boldsymbol{f}_0 = AE\alpha T_0 \int_0^L [\frac{dN_1}{dx} \ \frac{dN_2}{dx} \ \frac{dN_3}{dx}]^T (1 + x/L)dx = \frac{AE\alpha T_0}{4}[-5 \quad -2 \ \ 7]^T$$

Boundary load vector, $\boldsymbol{f}_b = [-N_{x=0} \ \ 0 \ \ N_{x=L}]^T$
Finally we end up with

$$\frac{2AE}{L}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} u_1(=0) \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} -N_{x=0} \\ 0 \\ 0 \end{bmatrix} + \frac{AE\alpha T_0}{4}\begin{bmatrix} -5 \\ -2 \\ 7 \end{bmatrix} + \frac{bL}{4}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

which gives $u_2 = \frac{5L\alpha T_0}{8} + \frac{3bL^2}{8AE}$

Solution 9.2

a) Strong form

$$\frac{dq}{dx} - Q = 0$$

Multiplication by a weight function and integration over the region of interest results in

$$\int \frac{dv}{dx} q \; dx - [vq] + \int vQ \; dx = 0$$

Choose weight function $v = \boldsymbol{N}\boldsymbol{c} = \boldsymbol{c}^T\boldsymbol{N}^T$, $\frac{dv}{dx} = \boldsymbol{c}^T\boldsymbol{B}^T$ giving

$$\int \boldsymbol{B}^T q \; dx = [\boldsymbol{N}^T q] - \int \boldsymbol{N}^T Q \; dx$$

Fick's law $q = -D\frac{dc}{dx}$, together with the interpolation $\frac{dc}{dx} = \boldsymbol{B}\boldsymbol{a}$ results in

$$-\int D\boldsymbol{B}^T\boldsymbol{B} \; dx \; \boldsymbol{a} = [\boldsymbol{N}^T q] - \int \boldsymbol{N}^T Q \; dx \qquad \Rightarrow \qquad \boldsymbol{K}\boldsymbol{a} = \boldsymbol{f}$$

where

$$\boldsymbol{K} = \int D\boldsymbol{B}^T\boldsymbol{B} \; dx, \qquad \boldsymbol{f} = -[\boldsymbol{N}^T q] + \int \boldsymbol{N}^T Q \; dx$$

b) Linear element $\boldsymbol{B} = [-1 \; 1]/L$ results in

$$\boldsymbol{K}^e = \frac{1}{L^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \int_0^L D \; dx$$

i.e.

$$\boldsymbol{K}^e_1 = \frac{D_0(L + aL^2)}{L^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \qquad \boldsymbol{K}^e_2 = \frac{D_0(L + 3aL^2)}{L^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Resulting in

$$\frac{D_0}{L}\begin{bmatrix} 1 + aL & -(1 + aL) & 0 \\ -(1 + aL) & 2 + 4aL & -(1 + 3aL) \\ 0 & -(1 + 3aL) & 1 + 3aL \end{bmatrix}\begin{bmatrix} c_m \\ c_2 \\ c_a \end{bmatrix} = \begin{bmatrix} f_1 \\ 0 \\ f_3 \end{bmatrix}$$

Row 2 gives

$$c_2 = \frac{(1 + aL)c_m + (1 + 3aL)c_a}{2 + 4aL}$$

c)

$$\boldsymbol{f}_b = -[\boldsymbol{N}^T q]_0^{2L} = -[\boldsymbol{N}^T q]_{x=2L} + [\boldsymbol{N}^T q]_{x=0} \quad \Rightarrow \quad \boldsymbol{f}_b = -[k\boldsymbol{N}^T\boldsymbol{N}]_{x=2L} \; \boldsymbol{a} + [\boldsymbol{N}^T q]_{x=0}$$

i.e.

$$\left(\boldsymbol{K} + k\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\boldsymbol{a} = [\boldsymbol{N}^T q]_{x=0}$$

Solution 9.3

a)

$$\int_0^L \frac{dv}{dx}\left(x\frac{dT}{dx}\right)dx + \int_0^L vN^2 T dx = \left[vx\frac{dT}{dx}\right]_0^L + \int_0^L vN^2 T_\infty dx$$

b)

$$\boldsymbol{K}\boldsymbol{a} = \boldsymbol{f}, \quad \text{where} \quad \boldsymbol{K} = \int_0^L \left(\boldsymbol{B}^T x \boldsymbol{B} + \boldsymbol{N}^T N^2 \boldsymbol{N}\right) dx \quad \text{and}$$

$$\boldsymbol{f} = \left(x\boldsymbol{N}^T \frac{dT}{dx}\right)|_{x=l} + \int_0^L \boldsymbol{N}^T N^2 T_\infty dx$$

c)

$$\frac{1}{2}\begin{bmatrix} 3 & 0 & 0 \\ 0 & 8 & -2 \\ 0 & -2 & 5 \end{bmatrix}\begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ (x\frac{dT}{dx})|_{x=l} \end{bmatrix} + N^2 T_\infty \frac{L}{4}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

d)

$$T = \boldsymbol{N}\boldsymbol{a} \quad \text{where} \quad \boldsymbol{a} = [0 \ 25 \ 100]^T$$

Solution 9.4 Multiplication and integration over the body and also use of the $c = \boldsymbol{N}\boldsymbol{a}$ as well as $v = \boldsymbol{N}\boldsymbol{c}$ results in

$$\left[\boldsymbol{N}^T k_1 \frac{dc}{dx}\right]_0^L - \int_0^L \boldsymbol{B}^T k_1 \boldsymbol{B} dx \boldsymbol{a} + \int_0^L \boldsymbol{N}^T Q dx = \int_0^L \boldsymbol{N}^T k_2 \boldsymbol{N} dx \dot{\boldsymbol{a}}$$

The element shape functions are given by $N_1^e = 1 - 2x/L$ and $N_2^e = 2x/L$. These shape functions results in $\boldsymbol{B}^e = [-2/L, \ \ 2/L]$. The following element functions can be obtained

$$\boldsymbol{K}_1^e = \int_0^{L/2} \boldsymbol{B}^{eT} k_1 \boldsymbol{B}^e dx = \frac{2k_1}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad \boldsymbol{C}_1^e = \int_0^{L/2} \boldsymbol{N}^{eT} k_2 \boldsymbol{N}^e dx = \frac{k_2 L}{12}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\boldsymbol{f}_{L1}^e = \int_0^{L/2} \boldsymbol{N}^{eT} Q dx = \frac{QL}{4}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The element matrices for the second element is identical to the element matrices of element 1. Assembly of the element matrices yields

$$\boldsymbol{K} = \int_0^L \boldsymbol{B}^T k_1 \boldsymbol{B} dx = \frac{2k_1}{L}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}, \quad \boldsymbol{C} = \int_0^L \boldsymbol{N}^T k_2 \boldsymbol{N} dx = \frac{k_2 L}{12}\begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

and

$$\boldsymbol{f}_L = \int_0^L \boldsymbol{N}^T Q dx = \frac{QL}{4}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad \boldsymbol{f}_b = \left[\boldsymbol{N}^T k_1 \frac{dc}{dx}\right]_0^L = \begin{bmatrix} k_1 q_0 \\ 0 \\ k_1(\frac{dc}{dx})|_{x=L} \end{bmatrix}, \quad \boldsymbol{a}(t=0) = \begin{bmatrix} 0 \\ c_0 \\ 0 \end{bmatrix}$$

The system can be written as $\boldsymbol{C}\dot{\boldsymbol{a}} + \boldsymbol{K}\boldsymbol{a} = \boldsymbol{f}_b + \boldsymbol{f}_L$.

Solution 9.5

- Weak form:

$$\left[vEA\frac{du}{dx}\right]_0^L - \int_0^L \frac{dv}{dx}EA\frac{du}{dx}dx + \int_0^L vbdx = \int_0^L vm\ddot{u}dx$$

- Finite element formulation:

$$\left[\boldsymbol{N}^T EA\frac{du}{dx}\right]_0^L - \int_0^L \boldsymbol{B}^T EA\boldsymbol{B}dx\boldsymbol{a} + \int_0^L \boldsymbol{N}^T bdx = \int_0^L \boldsymbol{N}^T m\boldsymbol{N}dx\ddot{\boldsymbol{a}}$$

- $\boldsymbol{K}$ is positive definite since

$$\boldsymbol{a}^T\boldsymbol{K}\boldsymbol{a} = \boldsymbol{a}^T\int_0^L \boldsymbol{B}^T EA\boldsymbol{B}dx\boldsymbol{a} = \int_0^L \boldsymbol{a}^T\boldsymbol{B}^T EA\boldsymbol{B}\boldsymbol{a}dx = \int_0^L EA\left(\frac{du}{dx}\right)^2 dx \geq 0$$

- System matrices:

$$\boldsymbol{K}^e = \frac{3AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad \boldsymbol{K} = \frac{3AE}{L}\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$$\boldsymbol{f}_l^e = \frac{bL}{6}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \boldsymbol{f}_l = \frac{bL}{6}\begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}$$

$$\boldsymbol{M}^e = \frac{mL}{18}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad \boldsymbol{M} = \frac{mL}{18}\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

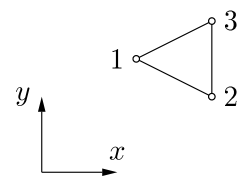
$$\left[\boldsymbol{N}^T EA\frac{du}{dx}\right]_0^L = \begin{bmatrix} -(EA\frac{du}{dx})|_{x=0} \\ 0 \\ 0 \\ (EA\frac{du}{dx})|_{x=L} \end{bmatrix}$$

10 FE formulation of 3D heat flow

Solution 10.1

a) Consult course book, pp. 206-208

b)



The approximation is given by $T = \alpha_1 + \alpha_2 x + \alpha_3 y$ which can be written as $T = [1\ x\ y][\alpha_1\ \alpha_2\ \alpha_3]^T = \bar{\boldsymbol{N}}\bar{\boldsymbol{\alpha}}$. Using the $C-matrix$ method we obtain

$$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} 1 & 850 & 750 \\ 1 & 1000 & 700 \\ 1 & 1000 & 800 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$$

The interpolation is obtained as $T = \bar{\boldsymbol{N}}\boldsymbol{C}^{-1}\boldsymbol{a}^e = \boldsymbol{N}^e\boldsymbol{a}^e$. The stiffness matrix is given by $\boldsymbol{K}^e = \int_{A_e} \boldsymbol{B}^{eT}\boldsymbol{D}\boldsymbol{B}^e dA$.

$$\boldsymbol{K}^e = \int_{A_e} \boldsymbol{B}^{eT}\boldsymbol{D}\boldsymbol{B}^e dA = A_e \boldsymbol{B}^{eT}\boldsymbol{B}^e k = kA_e \begin{bmatrix} \frac{-1}{150} & 0 \\ \frac{1}{300} & \frac{-1}{100} \\ \frac{1}{300} & \frac{1}{100} \end{bmatrix} \begin{bmatrix} \frac{-1}{150} & \frac{1}{300} & \frac{1}{300} \\ 0 & \frac{-1}{100} & \frac{1}{100} \end{bmatrix} =$$

$$k \begin{bmatrix} \frac{1}{3} & -\frac{1}{6} & -\frac{1}{6} \\ -\frac{1}{6} & \frac{5}{6} & -\frac{2}{3} \\ -\frac{1}{6} & -\frac{2}{3} & \frac{5}{6} \end{bmatrix}$$

Boundary term:

$$\int_{\mathcal{L}_{23}} \boldsymbol{N}^{eT} q_n d\mathcal{L} = \int_{\mathcal{L}_{23}} \boldsymbol{N}^{eT}\alpha(T - T_\infty) d\mathcal{L} = \int_{\mathcal{L}_{23}} \boldsymbol{N}^{eT}\alpha(\boldsymbol{N}^e\boldsymbol{a}^e - T_\infty) d\mathcal{L} =$$

$$\int_{\mathcal{L}_{23}} \alpha\boldsymbol{N}^{eT}\boldsymbol{N}^e d\mathcal{L}\boldsymbol{a}^e - \int_{\mathcal{L}_{23}} \alpha\boldsymbol{N}^{eT}T_\infty d\mathcal{L} = \frac{\alpha L}{6}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}\boldsymbol{a}^e - \alpha T_\infty L \begin{bmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

where $T_\infty = 22^o C$

Solution 10.2

a) Consult course book, pp. 220-221. Change $\boldsymbol{q}$ to $\boldsymbol{j}$ and T to V.
The result is $\int_A (\boldsymbol{\nabla} v)^T \sigma (\boldsymbol{\nabla} V) dA = -\oint_{\mathcal{L}} v \boldsymbol{j}^T \boldsymbol{n} d\mathcal{L}$ where v represents the weight functions.

Using the Galerkin choice of weight function we obtain

$$\int_A \boldsymbol{B}^T \sigma \boldsymbol{B} dA \boldsymbol{a} = -\oint_{\mathcal{L}} \boldsymbol{N}^T \boldsymbol{j}^T \boldsymbol{n} d\mathcal{L}$$

b)

$$\boldsymbol{a}^T \boldsymbol{K} \boldsymbol{a} = \int_A ||\boldsymbol{\nabla} V||^2 \sigma dA \geq 0$$

Find one $\boldsymbol{a} \neq \boldsymbol{0}$ such that $\int_A ||\boldsymbol{\nabla} V||^2 \sigma dA = 0$. Chose for instance $\boldsymbol{a} = [1 \quad 1 \quad ..... \quad 1 \quad 1]$.

Solution 10.3

a) Parasitic terms: $\alpha_7 x^2 y,\ \alpha_8 x y^2,\ \alpha_9 x^2 y^2$

b) The interpolation is complete and compatible, i.e. convergence is guaranteed.

c) Contribution to node 1, 2 and 3: $q_0/3,\ 4q_0/3,\ 2q_0/3$

Solution 10.4

FE-formulation $\int_A \boldsymbol{B}^T \boldsymbol{B} dA \boldsymbol{a} = \oint_{\mathcal{L}} \boldsymbol{N}^T (\boldsymbol{\nabla} c)^T \boldsymbol{n} d\mathcal{L}$

Concentration $\boldsymbol{a} = [1 \;\; 1 \;\; 2 \;\; 3/2]^T$, i.e. $a_3 = 2$ and $a_4 = 3/2$.

11 Guidelines for element meshes and global nodal nubering

12 Stresses and strains

Solution 12.1

a) $\boldsymbol{\epsilon} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$ Rigid body motion

b) $\boldsymbol{\epsilon} = \begin{bmatrix} 0 & k_1 & 0 \end{bmatrix}$ Uniaxial straining

c) $\boldsymbol{\epsilon} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$ Rigid body rotation

c) $\boldsymbol{\epsilon} = \begin{bmatrix} 0 & 0 & 2k_1 \end{bmatrix}$ Shear state

Solution 12.2

a) Global equilibrium

$$\int_S \boldsymbol{t} ds + \int_V \boldsymbol{b} dV = \mathbf{0}$$

Cauchys formula $\boldsymbol{t} = \boldsymbol{S}^T \boldsymbol{n}$ along with Gauss's theorem yields

$$\int_V (div(\boldsymbol{\sigma}^T) + \boldsymbol{b}) dV = \mathbf{0}$$

This balance should hold for arbitrary regions, i.e.

$$div(\boldsymbol{\sigma}^T) + \boldsymbol{b} = \mathbf{0}$$

b) The normal vector is given by

$$\boldsymbol{n} = \frac{1}{\sqrt{2}} [1\ 1\ 0]^T$$

which results in

$$\boldsymbol{t} = \frac{1}{\sqrt{2}} [18\ 2\ 0]^T$$

c)

$$\sigma_{nn} = 10 \quad \sigma_{nm} = 8$$

13 Linear elasticity

Solution 13.1 For the plane stress situation the strain energy is given as

$$W = 1/2[\sigma_{xx} \quad \sigma_{yy} \quad 0 \quad \sigma_{xy}][\epsilon_{xx} \quad \epsilon_{yy} \quad \epsilon_{zz} \quad \gamma_{xy}]^T = 1/2(\sigma_{xx}\epsilon_{xx} + \sigma_{yy}\epsilon_{yy} + \sigma_{xy}\gamma_{xy})$$

i.e. the ϵ_{zz} component does not contribute to the strain energy.

Solution 13.2 Consult course book pp. 254-256

14 FE formulation of non-circular shafts

15 Approximating functions for the FE-method-vector problems

Solution 15.1 Consult the course book, pp 282-286

16 FE formulation of three dimensional elasticity

Solution 16.1

a) The components of the displacement field are interpolated as $u_x = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy$ and $u_y = \beta_1 + \beta_2 x + \beta_3 y + \beta_4 xy$.

b) $\boldsymbol{B}$ $[3x8]$, $\boldsymbol{\sigma}$ $[3x1]$ $\boldsymbol{N}$ $[2x8]$, $\boldsymbol{b}$ $[2x1]$ $\boldsymbol{t}$ $[2x1]$

c)

$$\int_A \boldsymbol{B}^T \boldsymbol{\sigma} t dA = \int_A \boldsymbol{B}^T \boldsymbol{D}\boldsymbol{B} t dA \boldsymbol{a} - \int_A \boldsymbol{B}^T \boldsymbol{D}\boldsymbol{\epsilon}^\theta t dA$$

i.e.

$$\int_A \mathbf{B}^T \boldsymbol{D}\boldsymbol{B} t dA \boldsymbol{a} = \int_A \mathbf{N}^T \mathbf{b} t dA + \int_{\mathcal{L}} \mathbf{N}^T \mathbf{t} t d\mathcal{L} + \int_A \boldsymbol{B}^T \boldsymbol{D}\boldsymbol{\epsilon}^\theta t dA$$

Solution 16.2

a)
- $(t_x, t_y) = (0, 0)$ along $\mathcal{L}_1$
- $(t_x, t_y) = (ay, 0)$ along $\mathcal{L}_2$
- $(t_x, t_y) = (0, -ku_y)$ along $\mathcal{L}_3$
- $(t_x, t_y) = (0, 0)$ along $\mathcal{L}_4$
- $(u_x, u_y) = (0, 0)$ along $\mathcal{L}_5$

b) Consult the course book pp. 295-296

$$\int_A \boldsymbol{B}^T \boldsymbol{D}\boldsymbol{B} t dA \boldsymbol{a} = \int_{\mathcal{L}} \boldsymbol{N}^T \boldsymbol{t} t d\mathcal{L} + \int_A \boldsymbol{N}^T \boldsymbol{b} t dA$$

or

$$\int_A \boldsymbol{B}^T \boldsymbol{D}\boldsymbol{B} t dA \boldsymbol{a} = \int_{\mathcal{L}_3} \boldsymbol{N}^T \boldsymbol{t} t d\mathcal{L} + \int_{\mathcal{L}_{1,2,4,5}} \boldsymbol{N}^T \boldsymbol{t} t d\mathcal{L} + \int_A \boldsymbol{N}^T \boldsymbol{b} t dA$$

The term $\int_{\mathcal{L}_3} \boldsymbol{N}^T \boldsymbol{t} t d\mathcal{L}$ can be written as

$$\int_{\mathcal{L}_3} \boldsymbol{N}^T \boldsymbol{t} t d\mathcal{L} = -\int_{\mathcal{L}_3} \boldsymbol{N}^T \boldsymbol{k}\boldsymbol{N}\boldsymbol{a} t d\mathcal{L}$$

where $\boldsymbol{k} = \begin{bmatrix} 0 & 0 \\ 0 & k \end{bmatrix}$

c) Resulting stiffness matrix

$$\boldsymbol{K} = \int_A \boldsymbol{B}^T \boldsymbol{D}\boldsymbol{B} t dA + \int_{\mathcal{L}_3} \boldsymbol{N}^T \boldsymbol{k}\boldsymbol{N} t d\mathcal{L}$$

which due to the symmetry in $\boldsymbol{D}$ and $\boldsymbol{k}$ is symmetric.

d)

- $\boldsymbol{K} : [ndof \times ndof]$
- $\boldsymbol{a} : [ndof \times 1]$
- $\boldsymbol{B}^e : [3 \times 6]$
- $\boldsymbol{f} : [ndof \times 1]$
- $\boldsymbol{b} : [2 \times 1]$
- $\boldsymbol{K}^e : [6 \times 6]$
- $\boldsymbol{N}^e : [2 \times 6]$
- $\boldsymbol{D} : [3 \times 3]$
- $\boldsymbol{f}^e : [6 \times 1]$
- $\boldsymbol{t} : [2 \times 1]$

Solution 16.3

$$
\begin{bmatrix}
x & x & x & x & x & x & 0 & 0 & x & x & 0 & 0 & 0 & 0 & 0 & 0 \\
x & x & x & x & x & x & 0 & 0 & x & x & 0 & 0 & 0 & 0 & 0 & 0 \\
x & x & x & x & x & x & 0 & 0 & x & x & 0 & 0 & 0 & 0 & 0 & 0 \\
x & x & x & x & x & x & 0 & 0 & x & x & 0 & 0 & 0 & 0 & 0 & 0 \\
x & x & x & x & x & x & x & x & x & x & x & x & 0 & 0 & 0 & 0 \\
x & x & x & x & x & x & x & x & x & x & x & x & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & x & x & x & x & x & x & x & x & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & x & x & x & x & x & x & x & x & 0 & 0 & 0 & 0 \\
x & x & x & x & x & x & x & x & x & x & x & x & x & x & x & x \\
x & x & x & x & x & x & x & x & x & x & x & x & x & x & x & x \\
0 & 0 & 0 & 0 & x & x & x & x & x & x & x & x & x & x & x & x \\
0 & 0 & 0 & 0 & x & x & x & x & x & x & x & x & x & x & x & x \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x & x & x & x & x & x & x & x \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x & x & x & x & x & x & x & x \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x & x & x & x & x & x & x & x \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x & x & x & x & x & x & x & x
\end{bmatrix}
\begin{bmatrix} ? \\ 0 \\ 0 \\ 0 \\ ? \\ ? \\ ? \\ ? \\ ? \\ ? \\ ? \\ ? \\ ? \\ 0 \\ ? \\ ? \end{bmatrix}
=
\begin{bmatrix} 0 \\ ? \\ ? \\ ? \\ x \\ 0 \\ x \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ ? \\ 0 \\ 0 \end{bmatrix}
+
\begin{bmatrix} x \\ 0 \\ x \\ 0 \\ x \\ x \\ 0 \\ x \\ x \\ x \\ 0 \\ x \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

Solution 16.4 a)

$$\int_A \boldsymbol{B}^T \boldsymbol{\sigma} dA = \int_{\mathcal{L}} \boldsymbol{N}^T \boldsymbol{t} d\mathcal{L} + \int_A \boldsymbol{N}^T \boldsymbol{b} dA$$

Insertion of the constitutive law results in

$$\int_A \boldsymbol{B}^T \boldsymbol{D} \boldsymbol{B} dA \boldsymbol{a} = \int_V \boldsymbol{B}^T \boldsymbol{D} \boldsymbol{\epsilon}^{\Delta T} dA + \int_{\mathcal{L}} \boldsymbol{N}^T \boldsymbol{t} d\mathcal{L} + \int_A \boldsymbol{N}^T \boldsymbol{b} dA$$

Using that $t_x = -k_x u_x$ and $t_y = 0$ along L_2 we have

$$\begin{bmatrix} t_x \\ t_y \end{bmatrix} = - \begin{bmatrix} k_x & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix} \quad or \quad \boldsymbol{t} = -\boldsymbol{k}_2 \boldsymbol{u}$$

In the same way for L_3 we have

$$\begin{bmatrix} t_x \\ t_y \end{bmatrix} = - \begin{bmatrix} 0 & 0 \\ 0 & k_y \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix} \quad or \quad \boldsymbol{t} = -\boldsymbol{k}_3 \boldsymbol{u}$$

Using that $\boldsymbol{u} = \boldsymbol{N}\boldsymbol{a}$ we end up with

$$\left(\int_A \boldsymbol{B}^T \boldsymbol{D} \boldsymbol{B} dA + \int_{\mathcal{L}_2} \boldsymbol{N}^T \boldsymbol{k}_2 \boldsymbol{N} d\mathcal{L} + \int_{\mathcal{L}_3} \boldsymbol{N}^T \boldsymbol{k}_3 \boldsymbol{N} d\mathcal{L} \right) \boldsymbol{a} = \int_A \boldsymbol{B}^T \boldsymbol{D} \boldsymbol{\epsilon}^{\Delta T} dA + \int_A \boldsymbol{N}^T \boldsymbol{b} dA$$

where $\boldsymbol{b} = [0 \;\; -g\rho]^T$ where ρ represents the density per area.

b) Consult the course book, pp. 282-283. See also pp. 124-125.

17 FE formulation of beams

Solution 17.1

a)

$$\frac{d^2M}{dx^2} + q = 0$$

b) See course book equations 17.2.

c) M and V are natural boundary conditions whereas w and $\frac{dw}{dx}$ are essential boundary conditions.

d) See course book equations (17.30)-(17.41).

e)

$$w = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \alpha_4 x^3$$

Solution 17.2 a) Consult the course book page 318-319. The essential boundary conditions are $w = 0$ and $\frac{dw}{dx} = 0$ at $x = 0$ and $x = 3L$.

b) See course book equations (17.30)-(17.41).

c) The global stiffness matrix and load vectors.

$$
\boldsymbol{K} = EI \begin{bmatrix} 12/8L^3 & 6/4L^2 & -12/8L^3 & 6/4L^2 & 0 & 0 \\ 6/4L^2 & 4/2L & -6/4L^2 & 2/2L & 0 & 0 \\ -12/8L^3 & -6/4L^2 & 12/8L^3 + 12/L^3 & -6/4L^2 + 6/L^2 & -12/L^3 & 6/L^2 \\ 6/4L^2 & 2/2L & -6/4L^2 + 6/L^2 & 4/2L + 4/L & -6/L^2 & 2/L \\ 0 & 0 & -12/L^3 & -6/L^2 & 12/L^3 & -6/L^2 \\ 0 & 0 & 6/L^2 & 2/L & -6/L^2 & 4/L \end{bmatrix}
$$

$$
\boldsymbol{f}_l = \begin{bmatrix} 0 \\ 0 \\ 0 \\ M \\ 0 \\ 0 \end{bmatrix} \quad \boldsymbol{f}_b = \begin{bmatrix} V_1 \\ M_1 \\ 0 \\ 0 \\ V_3 \\ M_3 \end{bmatrix}
$$

The resulting system of equations that has to be solved is

$$
EI \begin{bmatrix} 12/8L^3 + 12/L^3 & -6/4L^2 + 6/L^2 \\ -6/4L^2 + 6/L^2 & 4/2L + 4/L \end{bmatrix} \begin{bmatrix} u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 0 \\ M \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}
$$

With numerical values the solution will be

$$
u_3 = -0.01323, \quad u_4 = 0.07937
$$

The deflection at the point A, use the approximations.

$$
w|_{0.5} = \boldsymbol{N}|_{0.5}^T \boldsymbol{a} = N_3|_{0.5}(-0.01323) + N_4|_{0.5} 0.07937 = -0.0165
$$

18 FE formulation of plates

19 Isoparametric mapping

Solution 19.1

a) Chose for instance $T = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy$. This approximation cannot satisfy the compatibility requirement on boundary $3-4$.

b) Line (3-4). $T = \boldsymbol{N}^e \boldsymbol{a}^e$. Since η or (ξ) is constant along the boundaries the temperature variation along all boundaries can be written as $T = \alpha_1 + \alpha_2 \eta$, i.e. two constants, two parameters.

Solution 19.2

The $\boldsymbol{B}^e$ matrix can be expressed as

$$\boldsymbol{B}^e = (\boldsymbol{J}^T)^{-1} \begin{bmatrix} \dfrac{\partial \boldsymbol{N}^e}{\partial \xi} \\ \dfrac{\partial \boldsymbol{N}^e}{\partial \eta} \end{bmatrix}, \quad \boldsymbol{J} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial x}{\partial \eta} \\ \dfrac{\partial y}{\partial \xi} & \dfrac{\partial y}{\partial \eta} \end{bmatrix}$$

wher the components in the jacobian are given as

$$\frac{\partial x}{\partial \xi} = \sum_i \frac{\partial N_i^e}{\partial \xi} x_i, \quad \frac{\partial x}{\partial \eta} = \sum_i \frac{\partial N_i^e}{\partial \eta} x_i$$

$$\frac{\partial y}{\partial \xi} = \sum_i \frac{\partial N_i^e}{\partial \xi} y_i, \quad \frac{\partial y}{\partial \eta} = \sum_i \frac{\partial N_i^e}{\partial \eta} y_i$$

20 Numerical integration

Solution 20.1 Shape functions along the upper boundary:

$$N_3 = \frac{x(x-0.5)}{1 \cdot 0.5} = 2x^2 - x \qquad N_7 = \frac{x(x-1)}{0.5(-0.5)} = 4x - 4x^2$$

a) Exact integration:

$$f_{b3y} = -\int_0^1 N_3 q dx = -q\left[\frac{2}{3}x^3 - \frac{1}{2}x^2\right]_0^1 = -\frac{1}{6}q$$

$$f_{b7y} = -\int_0^1 N_7 q dx = -q\left[2x^2 - \frac{4}{3}x^3\right]_0^1 = -\frac{2}{3}q$$

b) Numerical integration:

Map the domain [0 1] onto [−1 1] and perform the transformation $\int_0^1 N_i q dx = \int_{-1}^1 N_i q \frac{dx}{d\xi} d\xi$. The mapping is given by $\xi = -1 + 2x$, i.e. $\frac{dx}{d\xi} = 1/2$.

Using two integration points then results in

$$f_{b3y} = -\int_{-1}^1 N_3 q \frac{dx}{d\xi} d\xi = -0.16667q$$

$$f_{b7y} = -\int_{-1}^1 N_7 q \frac{dx}{d\xi} d\xi = -0.66667q$$

Solution 20.2 The map purely scales the element, i.e. $\boldsymbol{J} = 2\boldsymbol{I}$. If this is not realized directly, the components in the Jacobian is simply given as

$$\frac{\partial x}{\partial \xi} = \frac{\partial \boldsymbol{N}^e}{\partial \xi}\boldsymbol{x}^e = \frac{1}{4}(\eta-1)10 - \frac{1}{4}(\eta-1)14 + \frac{1}{4}(\eta+1)14 - \frac{1}{4}(\eta+1)10 = 2$$

the other three components are given in the same manner. In conclusion

$$\boldsymbol{J} = 2\boldsymbol{I} \quad det(\boldsymbol{J}) = 4 \quad \boldsymbol{J}^{-1} = \frac{1}{2}\boldsymbol{I} \quad (\boldsymbol{J}^{-1})^T = \frac{1}{2}\boldsymbol{I}$$

The component we seek is given by

$$K_{11}^e = \int_{-1}^{1}\int_{-1}^{1}\left[\left(\frac{\partial N_1^e}{\partial \xi}\right)^2 + \left(\frac{\partial N_1^e}{\partial \eta}\right)^2\right] d\xi d\eta = \int_{-1}^{1}\int_{-1}^{1}\frac{1}{16}\left[(\eta-1)^2 + (\xi-1)^2\right] d\xi d\eta$$

Gauss integration yield

$$K_{11}^e = \frac{1}{16}\left(\left[\left\{\frac{1}{\sqrt{3}}-1\right\}^2 + \left\{\frac{1}{\sqrt{3}}-1\right\}^2\right] + \left[\left\{-\frac{1}{\sqrt{3}}-1\right\}^2 + \left\{\frac{1}{\sqrt{3}}-1\right\}^2\right]\right.$$
$$\left. + \left[\left\{\frac{-1}{\sqrt{3}}-1\right\}^2 + \left\{\frac{-1}{\sqrt{3}}-1\right\}^2\right] + \left[\left\{\frac{1}{\sqrt{3}}-1\right\}^2 + \left\{\frac{-1}{\sqrt{3}}-1\right\}^2\right]\right) = \frac{2}{3}$$

Solution 20.3 Use the variable $\xi = -1 + 2x$

$$I = \int_0^1 f(x)dx = \int_{-1}^{1} f(\xi)\frac{dx}{d\xi}d\xi = \frac{1}{2}\int_{-1}^{1} f(\xi)d\xi \approx$$

$$\frac{1}{2}\left(f(\xi = -1/\sqrt{3}) + f(\xi = 1/\sqrt{3})\right) = \frac{1}{2}\left(f(x = \frac{1-1/\sqrt{3}}{2}) + f(x = \frac{1+1/\sqrt{3}}{2})\right) = 2$$

Exact result

$$I = \int_0^1 f(x)dx = \left[\frac{x^2}{2} + x + x^3 - \frac{x^4}{2}\right]_0^1 = \frac{1}{2} + 1 + 1 - \frac{1}{2} = 2$$

The exact result is expected since Gauss integration integrates a polynomial of order $2n-1$ where n is the number of integration points.