THE FINITE ELEMENT METHOD 2011
Dept. of Solid Mechanics, FHL064

FINAL EXAMINATION: 2011-06-03

A maximum of 60 points can be achieved in this examination. To pass at least 30 points are required.

Permitted aid: Pocket calculator.

Problem 1: (12p)

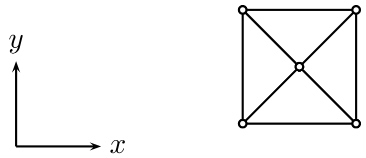


Figure 1: Mesh consisting of four 3-node plane elements.

The Laplace equation

$$div(\nabla\phi) = 0$$

should be solved using the finite element method in a square region with corners located at $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$.

a) Derive the weak form.

b) Derive the FE-formulation

c) Derive the stiffness matrix for the mesh depicted above.

Problem 2: (10p)

The finite element formulation for two dimensional heat conduction takes the form:

$$\left(\underbrace{\int_A \boldsymbol{B}^T\boldsymbol{D}\boldsymbol{B}dA}_{\boldsymbol{K}} + \underbrace{\int_{\mathcal{L}_c} \alpha \boldsymbol{N}^T\boldsymbol{N}d\mathcal{L}}_{\tilde{\boldsymbol{K}}}\right)\boldsymbol{a} = \underbrace{\int_{\mathcal{L}_c} \boldsymbol{N}^T\alpha T_\infty d\mathcal{L}}_{\boldsymbol{f}_c} \underbrace{- \int_{\mathcal{L}_g} \boldsymbol{N}^T q_n d\mathcal{L} - \int_{\mathcal{L}_h} \boldsymbol{N}^T h d\mathcal{L}}_{\boldsymbol{f}_b} + \underbrace{\int_A \boldsymbol{N}^T Q dA}_{\boldsymbol{f}_l}$$

where $\mathcal{L}_c$ is the boundary where convection applies, $\mathcal{L}_h$ is the boundary where the heat flow is prescribed and $\mathcal{L}_g$ is the boundary where the temperature is prescribed.

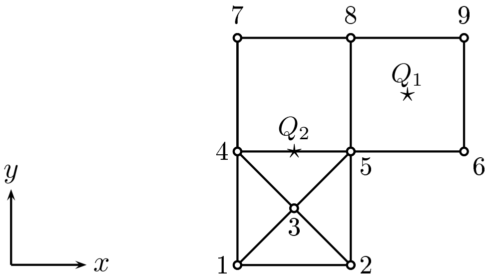


Figure 2: Mesh consisting of two four node elements and four three node elements.

The boundary conditions for the problem is given below

Prescribed temperature along node 1-2, i.e. $T = a$

Prescribed heat flow along 2-5-6-9, i.e. $q_n = h \neq 0$

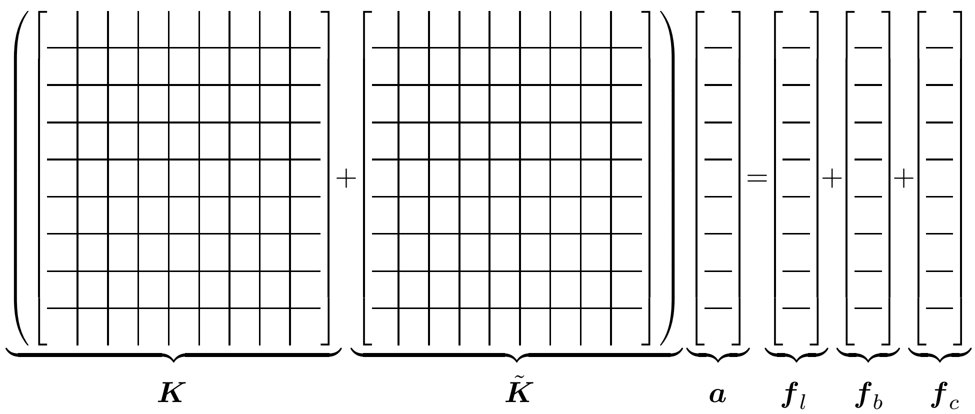
Insulation 9-8-7, i.e. $q_n = 0$

Convection along 7-4-1, i.e. $q_n = \alpha(T - T_\infty)$

Moreover, heat is supplied via the two point sources Q_1 and Q_2.

In the system of equations below mark

x – components that are known and different from zero
? – components that are unknown and different from zero
all blank positions are interpreted as zero.



Problem 3: (10p)

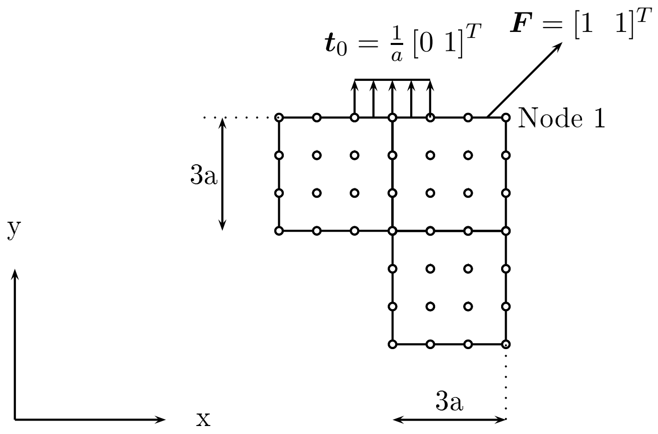


Figure 3: Mesh consisting of 16-node plane elements. For simplicity, only three elements are depicted.

The size of each element is $3a \times 3a$ and the nodes are located at equal distance. In a structural analysis the boundary force vector is given by

$$\boldsymbol{f}_b = \int_{\mathcal{L}} \boldsymbol{N}^T \boldsymbol{t} d\mathcal{L}$$

where the tickness was assumed to be 1. Calculate the contribution from the distributed force $\boldsymbol{t}_0$ and the point load $\boldsymbol{F}$ to node 1, i.e. f_{b1x} and f_{b1y}. The point load $\boldsymbol{F}$ is applied at an distance $a/2$ from node 1.

Problem 4: (10p)

The element stiffness relation in a one dimensional diffusion problem takes the following form

$$\boldsymbol{K}^e = \int_a^b \boldsymbol{B}^{eT} k(x) \boldsymbol{B}^e dx$$

where $\boldsymbol{B}^e = \frac{d}{dx}(\boldsymbol{N}^e)$.

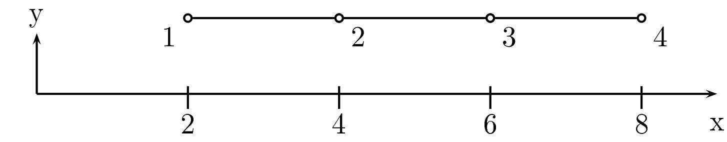


Figure 4: Finite element discretization

The diffusivity, $k(x)$ is given by

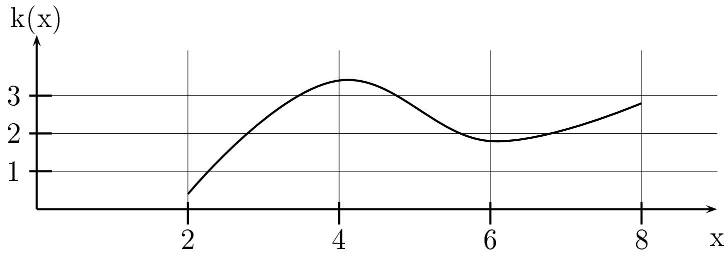


Figure 5: Variation of the diffusivity

Use (two point) gauss integration to calculate the element stiffness matrix for element 1, i.e. the element that extends from $x = 2$ to $x = 4$.

Hint: Estimate the conductuvity, $k(x)$ from the graph.

Problem 5: (8p)

The governing equations for a Bernoulli beam is given by

$$\frac{d^2M}{dx^2} + q = 0 \quad , \qquad M = -EI\frac{d^2w}{dx^2}$$

where q denotes the load intensity (positive in z-direction). Note that the equilibrium equation is derived from the following relations

$$\frac{dM}{dx} = V \quad , \qquad \frac{dV}{dx} = -q$$

a) Derive the weak form of the governing equation, and specify the essential and natural boundary conditions.
b) Derive the FE-formulation for the problem, such that a symmetric stiffness matrix is obtained.

Problem 6: (10p)

Consider the one dimensional differential equation

$$u'' + u + x = 0$$

along with the boundary conditions $u(0) = u(1) = 0$. To find an approximative solution to the differential equation, the following approximation is used

$$u^{app} = a_1 sin(\pi x) + a_2 x(x-1)$$

Use the weighted residual method $\int_0^1 v \cdot e dx = 0$ with the Galerkin method to obtain an approximative solution. v and e are the weight function and the residual, respectively.

Some useful integrals

$$\int_0^1 sin(\pi x)dx = 0.63662, \quad \int_0^1 sin^2(\pi x)dx = 0.50000$$

$$\int_0^1 x sin(\pi x)dx = 0.31831, \quad \int_0^1 x sin^2(\pi x)dx = 0.25000$$

$$\int_0^1 x^2 sin(\pi x)dx = 0.18930, \quad \int_0^1 x^2 sin^2(\pi x)dx = 0.14134$$

$$\int_0^1 x^k dx = 1/(k+1), \quad (k \neq -1)$$

Some hints that <u>might</u> be helpful

Hint: Green-Gauss's theorem states:

$$\int_A \phi div \boldsymbol{q} dA = \oint_{\mathcal{L}} \phi \boldsymbol{q}^T \boldsymbol{n} d\mathcal{L} - \int_A (\boldsymbol{\nabla}\phi)^T \boldsymbol{q} dA$$

Hint: Some trigonometric relations:

$$\sin(\alpha)^2 + \cos(\alpha)^2 = 1$$

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta), \quad \cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

$$\sin^2(\alpha) = \frac{1-\cos(2\alpha)}{2}, \quad \cos^2(\alpha) = \frac{1+\cos(2\alpha)}{2}$$

$$\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha), \quad \tan(2\alpha) = \frac{2\tan(\alpha)}{1-\tan^2(\alpha)}$$

Hint: A quadratic matrix is positive semidefinite if

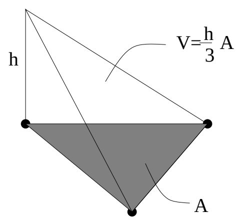
$$\boldsymbol{a}^T \boldsymbol{K} \boldsymbol{a} \geq 0, \quad \forall \boldsymbol{a}, \quad \text{and} \quad \boldsymbol{a}^T \boldsymbol{K} \boldsymbol{a} = 0 \quad \text{for some} \quad \boldsymbol{a} \neq \boldsymbol{0}$$

Hint: The interpolation formula of Lagrange is given by

$$l_k^{n-1} = \frac{(x-x_1).....(x-x_{k-1})(x-x_{k+1}).....(x-x_n)}{(x_k-x_1).....(x_k-x_{k-1})(x_k-x_{k+1}).....(x_k-x_n)}$$

Hint: Fourier's law is given by $\boldsymbol{q} = -\boldsymbol{D}\boldsymbol{\nabla}T$

Hint:



Hint: The position, ξ_i, of the integration points and weights, H_i, for n number of integration points can be found from

n	ξ_i	H_i
1	0	2
2	$\pm 1/\sqrt{3}$	1

where integration from -1 to 1 is assumed.

Hint:

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & a & b \\ 1 & c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} (ad-bc) & 0 & 0 \\ -d+b & d & -b \\ c-a & -c & a \end{bmatrix}$$