THE FINITE ELEMENT METHOD 2011
Dept. of Solid Mechanics, FHLF01

FINAL EXAMINATION: 2011-06-03

A maximum of 60 points can be achieved in this examination. To pass at least 30 points are required.

Permitted aid: Pocket calculator.

Problem 1: (12p)

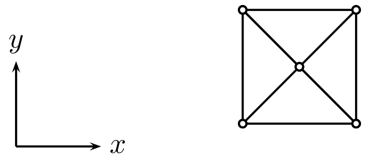


Figure 1: Mesh consisting of four 3-node plane elements.

The Laplace equation

$$div(\boldsymbol{\nabla}\phi) = 0$$

should be solved using the finite element method in a square region with corners located at $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$.

a) Derive the weak form.

b) Derive the FE-formulation

c) Derive the stiffness matrix for the mesh depicted above.

Problem 2: (10p)

The finite element formulation for two dimensional heat conduction takes the form:

$$\Bigg(\underbrace{\int_A \boldsymbol{B}^T\boldsymbol{D}\boldsymbol{B}dA}_{\boldsymbol{K}}+\underbrace{\int_{\mathcal{L}_c}\alpha\boldsymbol{N}^T\boldsymbol{N}d\mathcal{L}}_{\tilde{\boldsymbol{K}}}\Bigg)\boldsymbol{a}=\underbrace{\int_{\mathcal{L}_c}\boldsymbol{N}^T\alpha T_\infty d\mathcal{L}}_{\boldsymbol{f}_c}\underbrace{-\int_{\mathcal{L}_g}\boldsymbol{N}^T q_n d\mathcal{L}-\int_{\mathcal{L}_h}\boldsymbol{N}^T h d\mathcal{L}}_{\boldsymbol{f}_b}+\underbrace{\int_A \boldsymbol{N}^T Q dA}_{\boldsymbol{f}_l}$$

where $\mathcal{L}_c$ is the boundary where convection applies, $\mathcal{L}_h$ is the boundary where the heat flow is prescribed and $\mathcal{L}_g$ is the boundary where the temperature is prescribed.

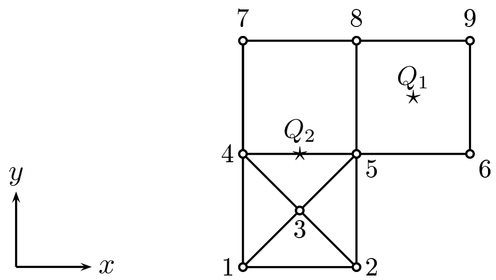


Figure 2: Mesh consisting of two four node elements and four three node elements.

The boundary conditions for the problem is given below

Prescribed temperature along node 1-2, i.e. $T = a$

Prescribed heat flow along 2-5-6-9, i.e. $q_n = h \neq 0$

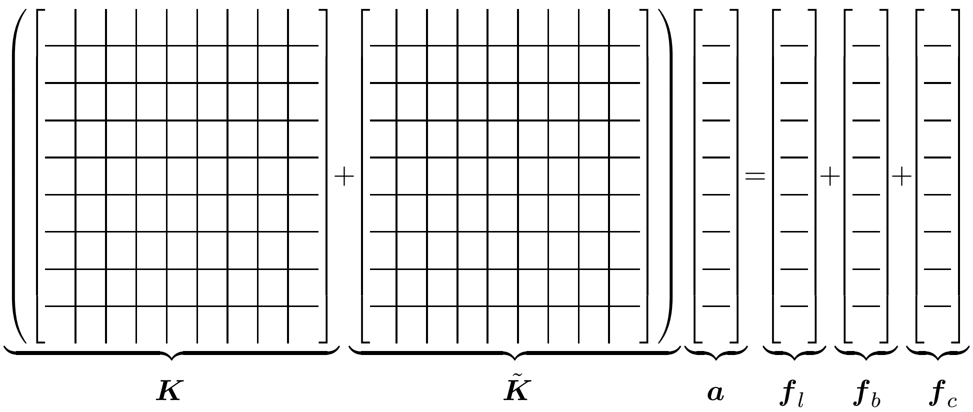
Insulation 9-8-7, i.e. $q_n = 0$

Convection along 7-4-1, i.e. $q_n = \alpha(T - T_\infty)$

Moreover, heat is supplied via the two point sources Q_1 and Q_2.

In the system of equations below mark

x – components that are known and different from zero
? – components that are unknown and different from zero
all blank positions are interpreted as zero.



Problem 3: (10p)

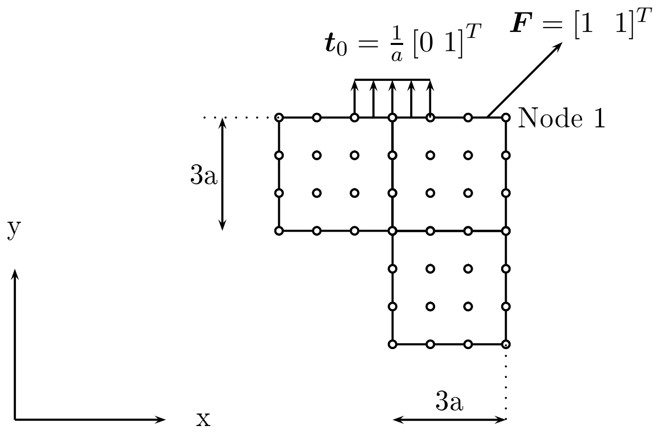


Figure 3: Mesh consisting of 16-node plane elements. For simplicity, only three elements are depicted.

The size of each element is $3a \times 3a$ and the nodes are located at equal distance. In a structural analysis the boundary force vector is given by

$$\boldsymbol{f}_b = \int_{\mathcal{L}} \boldsymbol{N}^T \boldsymbol{t} d\mathcal{L}$$

where the tickness was assumed to be 1. Calculate the contribution from the distributed force $\boldsymbol{t}_0$ and the point load $\boldsymbol{F}$ to node 1, i.e. f_{b1x} and f_{b1y}. The point load $\boldsymbol{F}$ is applied at an distance $a/2$ from node 1.

Problem 4: (10p)

The element stiffness relation in a one dimensional diffusion problem takes the following form

$$\boldsymbol{K}^e = \int_a^b \boldsymbol{B}^{eT} k(x) \boldsymbol{B}^e dx$$

where $\boldsymbol{B}^e = \frac{d}{dx}(\boldsymbol{N}^e)$.

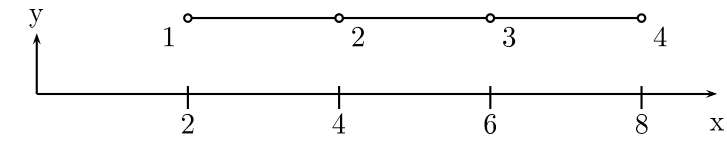


Figure 4: Finite element discretization

The diffusivity, $k(x)$ is given by

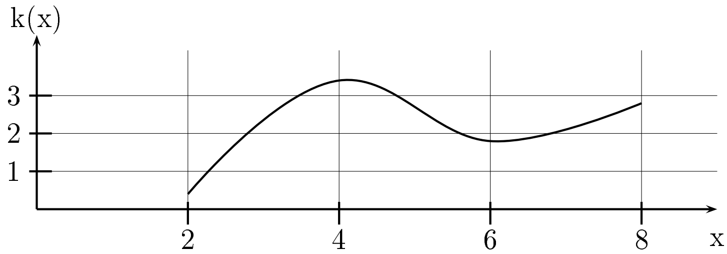
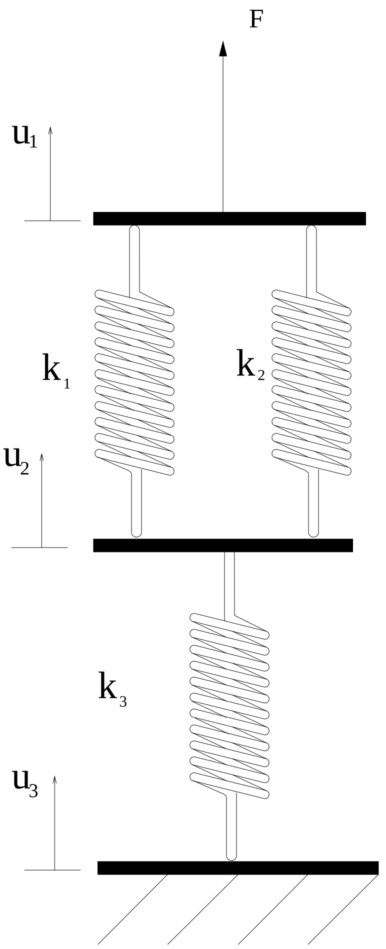


Figure 5: Variation of the diffusivity

Use (two point) gauss integration to calculate the element stiffness matrix for element 1, i.e. the element that extends from $x = 2$ to $x = 4$.

Hint: Estimate the conductuvity, $k(x)$ from the graph.

Problem 5: (8p)



a) Establish the strain energy for the system above.

b) Establish the potential for the system above.

c) Derive the equlibrium equations on the basis of minimization of the potential derived in b).

Problem 6: (10p)

Consider the one dimensional differential equation

$$u'' + u + x = 0$$

along with the boundary conditions $u(0) = u(1) = 0$. To find an approximative solution to the differential equation, the following approximation is used

$$u^{app} = a_1 sin(\pi x) + a_2 x(x-1)$$

Use the weighted residual method $\int_0^1 v \cdot e dx = 0$ with the Galerkin method to obtain an approximative solution. v and e are the weight function and the residual, respectively.

Some useful integrals

$$\int_0^1 sin(\pi x)dx = 0.63662, \quad \int_0^1 sin^2(\pi x)dx = 0.50000$$

$$\int_0^1 x sin(\pi x)dx = 0.31831, \quad \int_0^1 x sin^2(\pi x)dx = 0.25000$$

$$\int_0^1 x^2 sin(\pi x)dx = 0.18930, \quad \int_0^1 x^2 sin^2(\pi x)dx = 0.14134$$

$$\int_0^1 x^k dx = 1/(k+1), \quad (k \neq -1)$$

Some hints that <u>might</u> be helpful

Hint: Green-Gauss's theorem states:

$$\int_A \phi div \boldsymbol{q} dA = \oint_{\mathcal{L}} \phi \boldsymbol{q}^T \boldsymbol{n} d\mathcal{L} - \int_A (\boldsymbol{\nabla}\phi)^T \boldsymbol{q} dA$$

Hint: Some trigonometric relations:

$$\sin(\alpha)^2 + \cos(\alpha)^2 = 1$$

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta), \quad \cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

$$\sin^2(\alpha) = \frac{1-\cos(2\alpha)}{2}, \quad \cos^2(\alpha) = \frac{1+\cos(2\alpha)}{2}$$

$$\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha), \quad \tan(2\alpha) = \frac{2\tan(\alpha)}{1-\tan^2(\alpha)}$$

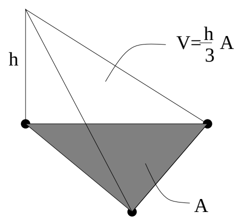
Hint: A quadratic matrix is positive semidefinite if

$$\boldsymbol{a}^T \boldsymbol{K} \boldsymbol{a} \geq 0, \quad \forall \boldsymbol{a}, \quad \text{and} \quad \boldsymbol{a}^T \boldsymbol{K} \boldsymbol{a} = 0 \quad \text{for some} \quad \boldsymbol{a} \neq \boldsymbol{0}$$

Hint: The interpolation formula of Lagrange is given by

$$l_k^{n-1} = \frac{(x-x_1).....(x-x_{k-1})(x-x_{k+1}).....(x-x_n)}{(x_k-x_1).....(x_k-x_{k-1})(x_k-x_{k+1}).....(x_k-x_n)}$$

Hint: Fourier's law is given by $\boldsymbol{q} = -\boldsymbol{D}\boldsymbol{\nabla}T$
Hint:



Hint: The position, ξ_i, of the integration points and weights, H_i, for n number of integration points can be found from

n	ξ_i	H_i
1	0	2
2	$\pm 1/\sqrt{3}$	1

where integration from -1 to 1 is assumed.

Hint:

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & a & b \\ 1 & c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} (ad-bc) & 0 & 0 \\ -d+b & d & -b \\ c-a & -c & a \end{bmatrix}$$