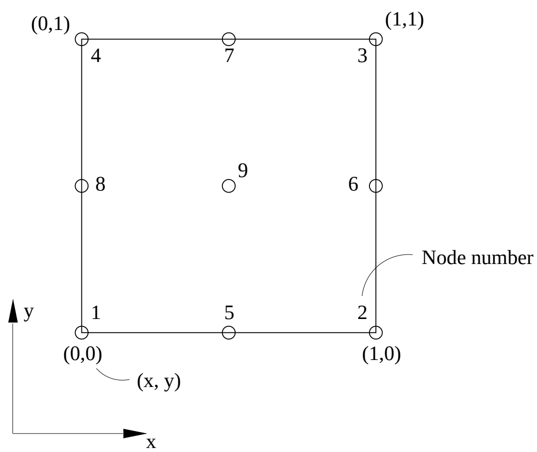
THE FINITE ELEMENT METHOD, FHL064 2012
Dept. of Solid Mechanics

FINAL EXAMINATION: 2012-05-31

A maximum of 60 points can be achieved in this examination. To pass at least 30 points are required.

Problem 1: (8p)



A nine-node Lagrange element have been used in a two-dimensional heat conduction FE-analysis. The solution for the element is given by

$$\boldsymbol{a}^T = [0.3\ 0.2\ 0.76\ 0.72\ 0.25\ 0.46\ 0.27\ 0.0\ 0.29]$$

- Calculate the temperature in point (0,0.25).

Problem 2: (12p)

Consider a box of dimensions $4 \times 4 \times 3$ with the eight corners located at $(\pm 2, \pm 2, -1)$ and $(\pm 2, \pm 2, 2)$. Use Gauss integration (use one integration point) to calculate the integral

$$I = \int_V f dV$$

where $f = x + y + z$.

Problem 3: (6p)

For a beam of length a the interpolation for the displacement is given by

$$w = N_1^e u_1 + N_2^e u_2 + N_3^e u_3 + N_4^e u_4$$

a) Sketch the shape functions

b) Indicate in all sketches if the shape function or slope of the shape function has the value 1 at the nodal points.

Problem 4: (12p) HAND IN THIS PAPER ! NAME:

The finite element formulation for two dimensional heat conduction takes the form:

$$\left(\underbrace{\int_A \boldsymbol{B}^T \boldsymbol{D} \boldsymbol{B} dA}_{\boldsymbol{K}} + \underbrace{\int_{\mathcal{L}_c} \alpha \boldsymbol{N}^T \boldsymbol{N} d\mathcal{L}}_{\tilde{\boldsymbol{K}}} \right) \boldsymbol{a} = \underbrace{\int_{\mathcal{L}_c} \boldsymbol{N}^T \alpha T_\infty d\mathcal{L}}_{\boldsymbol{f}_c} \underbrace{- \int_{\mathcal{L}_g} \boldsymbol{N}^T q_n d\mathcal{L} - \int_{\mathcal{L}_h} \boldsymbol{N}^T h d\mathcal{L}}_{\boldsymbol{f}_b} + \underbrace{\int_A \boldsymbol{N}^T Q dA}_{\boldsymbol{f}_l}$$

where $\mathcal{L}_c$ is the boundary where convection applies, $\mathcal{L}_h$ is the boundary where the heat flow is prescribed and $\mathcal{L}_g$ is the boundary where the temperature is prescribed.

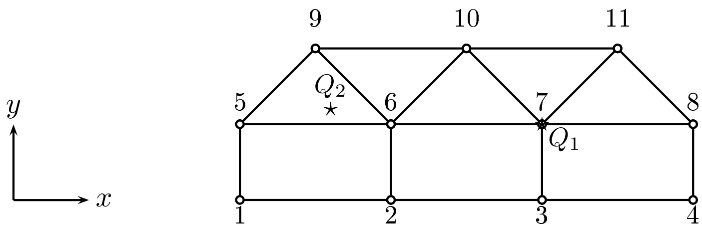


Figure 1: Mesh consisting of three four node elements and five three node elements.

The boundary conditions for the problem is given below

Prescribed temperature along nodes 9-10-11, i.e. $T = a$

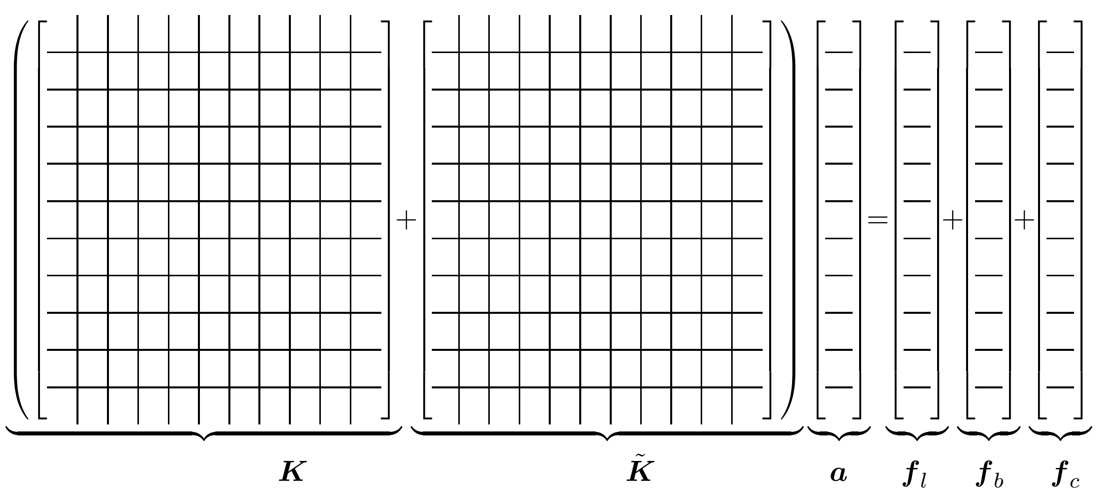
Prescribed heat flow along 1-2-3-4, i.e. $q_n = h \neq 0$

Convection along 1-5-9 and 4-8-11, i.e. $q_n = \alpha(T - T_\infty)$

Moreover, heat is supplied via the two point sources Q_1 and Q_2.

In the system of equations below mark

x – components that are known and different from zero
? – components that are unknown and different from zero
all blank positions are interpreted as zero.



Problem 5: (12p)

A two dimensional boundary value problem is governed by

$$\frac{\partial}{\partial x}\left((1+x^2)\frac{\partial \varphi}{\partial x}\right) + \frac{\partial}{\partial y}\left((1+y^2)\frac{\partial \varphi}{\partial y}\right) + \xi = \dot{\varphi}$$

where $\xi = \xi(x,y)$ is time independent.

a) Derive the weak form corresponding to the differential equation above.

b) Derive the finite element equation.

c) A part of a domain should be modelled using four node quadratic elements. Suggest an approximation for that element. (You may assume that the element sides are parallell with the coordinate axes.)

d) Does the proposed approximation fulfill the convergence criterion ?

e) Does the proposed approximation involve any parasitic terms ?

Problem 6: (10p)

The stiffness matrix and the load vector in a heat conduction problem reads

$$\boldsymbol{K} = \int_A \boldsymbol{B}^T \boldsymbol{D} \boldsymbol{B} dA, \quad \boldsymbol{f}_l = \int_A \boldsymbol{N}^T Q dA$$

For element 1 depicted below, calculate the element stiffness matrix and the load vector.

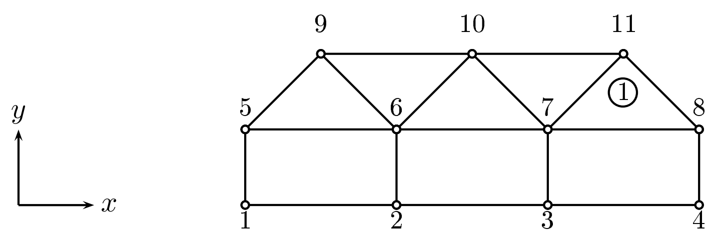


Figure 2: Mesh consisting of three four node elements and five three node elements.

The coordinates for node 7 is $(7,1)$, node 11 $(8,2)$ and node 8 $(9,1)$. The constitutive matrix $\boldsymbol{D}$ is given by $\boldsymbol{D} = k\boldsymbol{I}$ where the conductivity is known and constant. The heat source Q, is in this case a point source located at $(8, 1.75)$ with the magnitude Q_0.

Some hints that <u>might</u> be helpful

Hint: Green-Gauss's theorem states:

$$\int_A \phi \; div \boldsymbol{q} dA = \oint_{\mathcal{L}} \phi \boldsymbol{q}^T \boldsymbol{n} d\mathcal{L} - \int_A (\boldsymbol{\nabla}\phi)^T \boldsymbol{q} dA$$

Hint: Some trigonometric relations:

$$\sin^2(\alpha) + \cos^2(\alpha) = 1$$

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta), \quad \cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

$$\sin^2(\alpha) = \frac{1-\cos(2\alpha)}{2}, \quad \cos^2(\alpha) = \frac{1+\cos(2\alpha)}{2}$$

$$\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha), \quad \tan(2\alpha) = \frac{2\tan(\alpha)}{1-\tan^2(\alpha)}$$

Hint: A quadratic matrix is positive semidefinite if

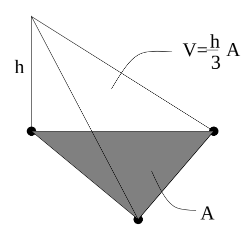
$$\boldsymbol{a}^T \boldsymbol{K} \boldsymbol{a} \geq 0, \quad \forall \boldsymbol{a}, \quad \text{and} \quad \boldsymbol{a}^T \boldsymbol{K} \boldsymbol{a} = 0 \quad \text{for some} \quad \boldsymbol{a} \neq \boldsymbol{0}$$

Hint: The interpolation formula of Lagrange is given by

$$l_k^{n-1} = \frac{(x-x_1).....(x-x_{k-1})(x-x_{k+1}).....(x-x_n)}{(x_k-x_1).....(x_k-x_{k-1})(x_k-x_{k+1}).....(x_k-x_n)}$$

Hint: Fourier's law is given by $\boldsymbol{q} = -\boldsymbol{D}\boldsymbol{\nabla}T$

Hint:



Hint: The position, ξ_i, of the integration points and weights, H_i, for n number of integration points can be found from

n	ξ_i	H_i
1	0	2
2	$\pm 1/\sqrt{3}$	1

where integration from -1 to 1 is assumed.

Hint:

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & a & b \\ 1 & c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} (ad-bc) & 0 & 0 \\ -d+b & d & -b \\ c-a & -c & a \end{bmatrix}$$