THE FINITE ELEMENT METHOD 2013
Dept. of Solid Mechanics

FINAL EXAMINATION: 2013-06-05

A maximum of 60 points can be achieved in this examination. To pass at least 30 points are required.

Permitted aid: Pocket calculator.

Problem 1 : (10p)

In phase-field based structural optimization, the differential equation below needs to be solved.

$$-\gamma\dot{\rho} = \rho + \gamma div(\boldsymbol{\nabla}\rho) - \lambda$$

In this evolution eqution the seeked field, ρ represents the optimal design. The variables λ and γ can be considered as constants. The boundary of the analyzed domain, V, is separeted into one part S_h where $\boldsymbol{\nabla}\rho \cdot \boldsymbol{n} = 0$ and one part S_c where $\gamma\boldsymbol{\nabla}\rho \cdot \boldsymbol{n} = (1 + \rho)$.

a) Derrive the weak form of the problem stated above.

b) Derrive the finite element formulation corresponding to the problem above. Note that you do not need to calculate any matrices.

Problem 2 : (10p)

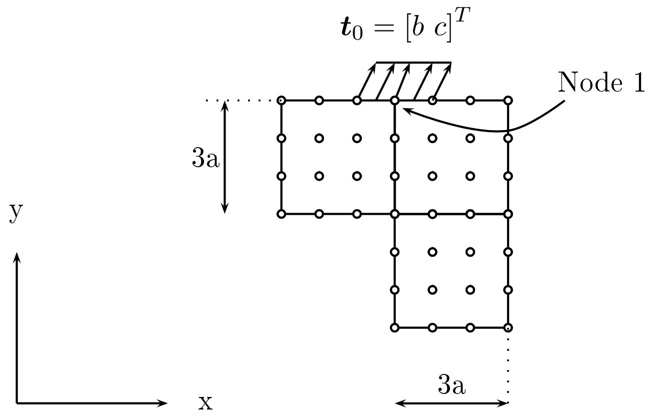


Figure 1: Mesh consisting of 16-node plane elements. For simplicity, only three elements are depicted.

The size of each element is $3a \times 3a$ and the nodes are located at equal distance. In a structural analysis the boundary force vector is given by

$$\boldsymbol{f}_b = \int_{\mathcal{L}} \boldsymbol{N}^T \boldsymbol{t} d\mathcal{L}$$

where the thickness was assumed to be 1. Calculate the contribution from the distributed force $\boldsymbol{t}_0$ to node 1, i.e. f_{b1x} and f_{b1y} using Gauss integration. You should use two integration points for the evaluation of $\boldsymbol{f}_b^e$

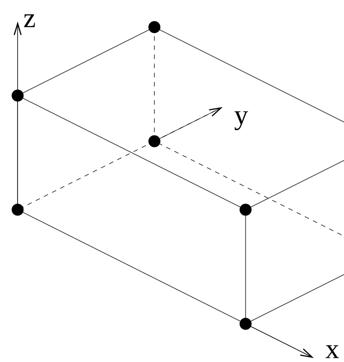
Problem 3 : (10p)

The finite element formulation for heat conduction takes the form:

$$\left(\underbrace{\int_V \boldsymbol{B}^T\boldsymbol{D}\boldsymbol{B}dV}_{\boldsymbol{K}} + \underbrace{\int_{S_c} \alpha\boldsymbol{N}^T\boldsymbol{N}dS}_{\tilde{\boldsymbol{K}}}\right)\boldsymbol{a} = \underbrace{\int_{S_c} \boldsymbol{N}^T\alpha T_\infty dS}_{\boldsymbol{f}_c} \underbrace{- \int_{S_g} \boldsymbol{N}^T q_n dS - \int_{S_h} \boldsymbol{N}^T h dS}_{\boldsymbol{f}_b} + \underbrace{\int_V \boldsymbol{N}^T Q dV}_{\boldsymbol{f}_l}$$

where S_c is the boundary where convection applies, S_h is the boundary where the heat flow is prescribed and S_g is the boundary where the temperature is prescribed.

The finite element model is based on brick elements as depicted below.

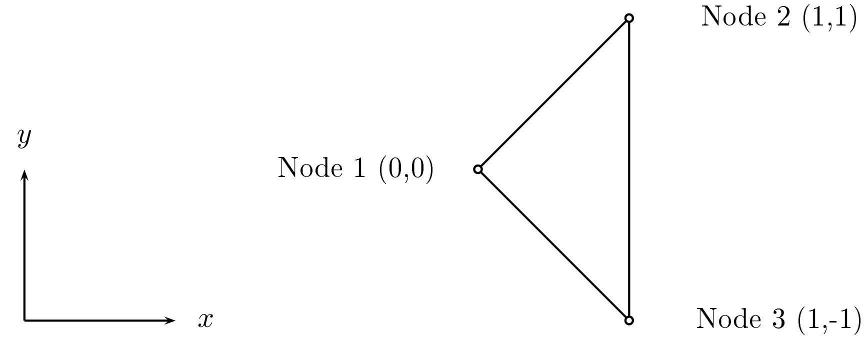


a) Show that the matrix $\boldsymbol{K} + \tilde{\boldsymbol{K}}$ is positive definite for $\alpha > 0$. $\boldsymbol{D}$ is symmetric and positive semi definite.

b) Suggest an approximation for the brick element that satisfies the convergence requirement. (Proof needed).

Problem 4 : (10p)

The element depicted below has been used in a thermo-mechanical anaysis.



The result of the analysis is

$$\begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix} = \begin{bmatrix} 1 \cdot 10^{-4} \\ 2 \cdot 10^{-4} \\ -1 \cdot 10^{-4} \\ 3 \cdot 10^{-4} \\ 0 \\ 0 \end{bmatrix} m$$

and

$$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} 140 \\ 180 \\ 250 \end{bmatrix} {}^\circ C$$

Calculate the stress distribution in the element using the Hooke's law (plane stress prevails). The stifness is assumed to be given by

$$\boldsymbol{D} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix}$$

The termal strains are given by $\boldsymbol{\epsilon}^o = \alpha \Delta T [1\ 1\ 0]^T$. The stress free temperature is $20^\circ C$ where the expansion coefficient is given by $\alpha = 4 \cdot 10^{-6}$. The Poisson's ratio is given by $\nu = 0.3$ and the Young's modulus is given by $E = 210 \cdot 10^3 MPa$

Problem 5 : (10p)

The finite element formulation for two dimensional heat conduction takes the form:

$$\left(\underbrace{\int_A \boldsymbol{B}^T\boldsymbol{D}\boldsymbol{B}dA}_{\boldsymbol{K}} + \underbrace{\int_{\mathcal{L}_c} \alpha\boldsymbol{N}^T\boldsymbol{N}d\mathcal{L}}_{\tilde{\boldsymbol{K}}}\right)\boldsymbol{a} = \underbrace{\int_{\mathcal{L}_c} \boldsymbol{N}^T\alpha T_\infty d\mathcal{L}}_{\boldsymbol{f}_c} \underbrace{- \int_{\mathcal{L}_g} \boldsymbol{N}^T q_n d\mathcal{L} - \int_{\mathcal{L}_h} \boldsymbol{N}^T h d\mathcal{L}}_{\boldsymbol{f}_b} + \underbrace{\int_A \boldsymbol{N}^T Q dA}_{\boldsymbol{f}_l}$$

where $\mathcal{L}_c$ is the boundary where convection applies, $\mathcal{L}_h$ is the boundary where the heat flow is prescribed and $\mathcal{L}_g$ is the boundary where the temperature is prescribed.

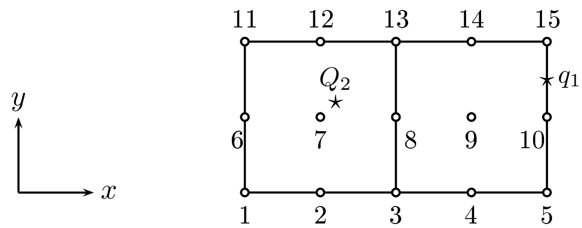


Figure 2: Mesh consisting of two nino-node elements.

The boundary conditions for the problem is given below

Prescribed temperature along nodes 1-2-3-4-5, i.e. $T = g$

Prescribed heat flow along 1-6-11, i.e. $q_n = h \neq 0$

Isolated along 5-10-15, except for the boundary point source q_1

Convection along 11-12-13-14-15, i.e. $q_n = \alpha(T - T_\infty)$

Moreover, heat is also supplied via the point source Q_2.

In the system of equations below mark

x – components that are known and different from zero. Assume that that the element sfiffness, $\boldsymbol{K}^e$ matrix is fully populated.
? – components that are unknown and different from zero
all blank positions are interpreted as zero.

$$\underbrace{\left(\left[\;\right]\right.}_{\boldsymbol{K}} + \underbrace{\left.\left[\;\right]\right)}_{\tilde{\boldsymbol{K}}} \underbrace{\left[\;\right]}_{\boldsymbol{a}} = \underbrace{\left[\;\right]}_{\boldsymbol{f}_l} + \underbrace{\left[\;\right]}_{\boldsymbol{f}_b} + \underbrace{\left[\;\right]}_{\boldsymbol{f}_c}$$

Problem 6 (FHL064) : (10p)

Equilibrium for a beam along with the Bernoulli beam theory can be summarized as

$$\frac{dV}{dx} = -q \quad ; \qquad \frac{dM}{dx} = V, \quad M = -E^* I^* \frac{d^2 w}{dx^2}$$

where q is the load per unit length, w the deflection, M is the bending moment and V is the shear force. $E^* I$ represents material and geometry dependent parameters which is assumed to solely depend on the coordinate, x.

a) From the relations above, establish the weak form.

b) Based on the weak form, establish the finite element formulation.

c) What are the natural and essential boundary conditions for a FE-formulation for Bernoulli beams ?

Problem 6 (FHLF01) : (10p)

The solution to the differential equation

$$-\text{div}(\nabla \phi) = 2(x + y) - 4$$

in a region Ω coincides with the minumum to

$$\Pi = \int_{\Omega} \left(\frac{1}{2} (\nabla \phi)^T \nabla \phi - \phi \left[2(x + y) - 4 \right] \right) dV$$

The region, Ω considered here is defined by it's corner nodes (0,0), (1,0), (1,1) och (0,1) and the boundary conditions are given by

$$\begin{aligned}
\phi(0, y) &= 0 \\
\phi(x, 0) &= 0 \\
\phi(1, y) &= 0 \\
\phi(x, 1) &= 0
\end{aligned}$$

a) Suggest an approximation that incolves two parameters that fulfills the boundary conditions.

b) Based on the minimization problem establish $\boldsymbol{K a} = \boldsymbol{f}$ where $\boldsymbol{a}$ is a vetor that contains the two unknown parameters. NOTE that no integrals have to be evaluated.

Some hints that <u>might</u> be helpful

Hint: Green-Gauss's theorem states:

$$\int_A \phi \; div\boldsymbol{q} dA = \oint_{\mathcal{L}} \phi \boldsymbol{q}^T \boldsymbol{n} d\mathcal{L} - \int_A (\boldsymbol{\nabla}\phi)^T \boldsymbol{q} dA$$

Hint: The strain components are defined as

$$\epsilon_{xx} = \frac{\partial u_x}{\partial x}, \; \epsilon_{xx} = \frac{\partial u_y}{\partial y}, \; \epsilon_{zz} = \frac{\partial u_z}{\partial z}$$

and

$$\gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}, \; \gamma_{xz} = \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}, \; \gamma_{yz} = \frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z},$$

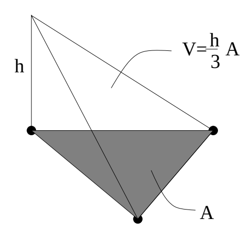
Hint: A quadratic matrix is positive semidefinite if

$$\boldsymbol{a}^T \boldsymbol{K} \boldsymbol{a} \geq 0, \quad \forall \boldsymbol{a}, \quad \text{and} \quad \boldsymbol{a}^T \boldsymbol{K} \boldsymbol{a} = 0 \quad \text{for some} \quad \boldsymbol{a} \neq \boldsymbol{0}$$

Hint: The interpolation formula of Lagrange is given by

$$l_k^{n-1} = \frac{(x-x_1).....(x-x_{k-1})(x-x_{k+1}).....(x-x_n)}{(x_k-x_1).....(x_k-x_{k-1})(x_k-x_{k+1}).....(x_k-x_n)}$$

Hint: Fourier's law is given by $\boldsymbol{q} = -\boldsymbol{D}\boldsymbol{\nabla}T$
Hint: Hooke's law for thermo-elasticity is given by $\boldsymbol{\sigma} = \boldsymbol{D}\left(\boldsymbol{\epsilon} - \boldsymbol{\epsilon}^o\right)$
Hint:



Hint: The position, ξ_i, of the integration points and weights, H_i, for n number of integration points can be found from

n	ξ_i	H_i
1	0	2
2	$-1/\sqrt{3}$, $1/\sqrt{3}$	1,1
3	0 , $\sqrt{3/5}$, $-\sqrt{3/5}$	8/9, 5/9 , 5/9

where integration from -1 to 1 is assumed.

Hint:

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & a & b \\ 1 & c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} (ad-bc) & 0 & 0 \\ -d+b & d & -b \\ c-a & -c & a \end{bmatrix}$$