THE FINITE ELEMENT METHOD 2014
Dept. of Solid Mechanics

FINAL EXAMINATION: 2014-05-31

A maximum of 60 points can be achieved in this examination. To pass at least 30 points are required.

Permitted aid: Pocket calculator.

Problem 1 : (5p)

The governing equation for a spring system takes the following form

$$\begin{bmatrix} 12 & -4 & -8 \\ -4 & 6 & -2 \\ -8 & -2 & 10 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix}$$

where u_1, u_2 and u_3 are the nodal displacement and F_1, F_2 and F_3 are the external forces acting at the nodal points. Note that no boundary conditions have been introduced.

a) Solve the system for the boundary conditions $u_1 = 1$, $F_2 = 2$ and $F_3 = 3$.

b) Based on physical arguments, explain why the system above can not be solved before any displacement boundary conditions are imposed.

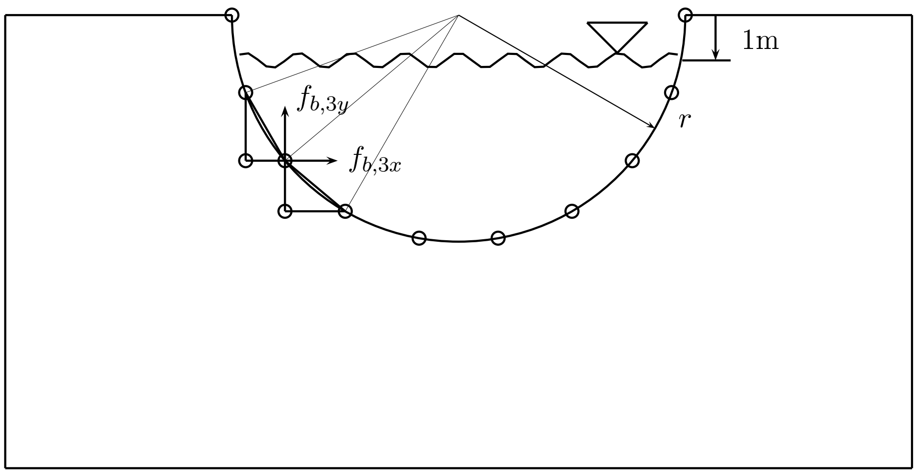
Problem 2 : (5p)

A finite element formulation of the heat conduction problem $div(\boldsymbol{q}) = Q$ results in the load vector

$$\boldsymbol{f} = \int_V \boldsymbol{N}^T Q dV - \int_S \boldsymbol{N}^T q_n dS$$

where $q_n = \boldsymbol{q}^T \boldsymbol{n}$. Show that $\Sigma_i (\boldsymbol{f})_i = 0$.

Problem 3 : (12p)

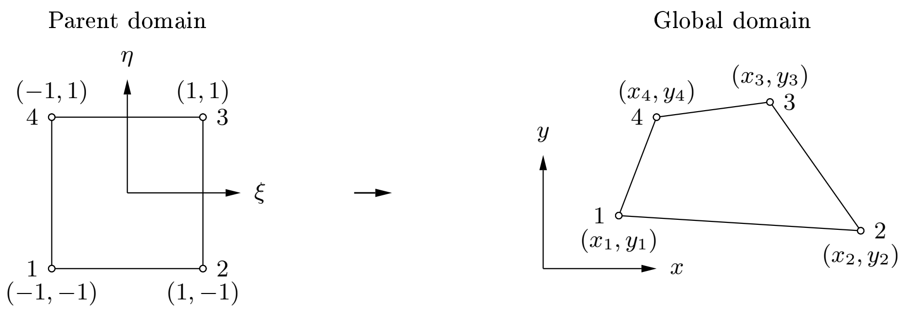


Consider the water channel depicted above. The stresses arising from the water pressure should be analyzed using the finite element method. One step in the finite element procedure consists of calculation of the vector $\boldsymbol{f}_b$ due to the boundary forces.

Determine the x-component of the boundary vector in node 3, i.e. $f_{b,3x}$. The x-axis is taken to be parallel with the horizontal axis. The circular shape boundary of the channel is split into 9 triangular 3-node elements. Each element corresponds to 20^o of the circle shape and the radius, r, of the channel is 15m. Moreover, the water level in the channel is 1m below the edge. The thickness, t, of the channel is 1m. The pressure, p, is given by $p = \rho g h$ where and ρ and g are known. The depth is given by h and please note that the pressure is acting perpendicular to the surface.

Hint. The boundary force term is defined by $\boldsymbol{f}_b = \int_{\mathcal{L}} t\boldsymbol{N}^T\boldsymbol{t}d\mathcal{L}$.

Problem 4 : (6p)



A temperature problem is to be analysed using iso-parametric elements and the approximation for the temperature field within an element is given by

$$T(\xi, \eta) = \boldsymbol{N}^e(\xi, \eta)\boldsymbol{a}^e$$

and the stiffness matrix is given by

$$\boldsymbol{K}^e = \int_A \boldsymbol{B}^T \boldsymbol{D}\boldsymbol{B} dA; \quad \boldsymbol{D} = k\boldsymbol{I}; \quad \boldsymbol{B} = \begin{bmatrix} \dfrac{\partial \boldsymbol{N}^e}{\partial x} \\ \dfrac{\partial \boldsymbol{N}^e}{\partial y} \end{bmatrix}$$

- How are the coordinates interpolated in an isoparametric element ?
- Show how the element stiffness matrix can be evaluated over the parent domain.

Hint: The following relation holds

$$\begin{aligned} \frac{\partial \boldsymbol{N}^e}{\partial \xi} &= \frac{\partial \boldsymbol{N}^e}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial \boldsymbol{N}^e}{\partial y}\frac{\partial y}{\partial \xi} \\ \frac{\partial \boldsymbol{N}^e}{\partial \eta} &= \frac{\partial \boldsymbol{N}^e}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial \boldsymbol{N}^e}{\partial y}\frac{\partial y}{\partial \eta} \end{aligned}$$

and

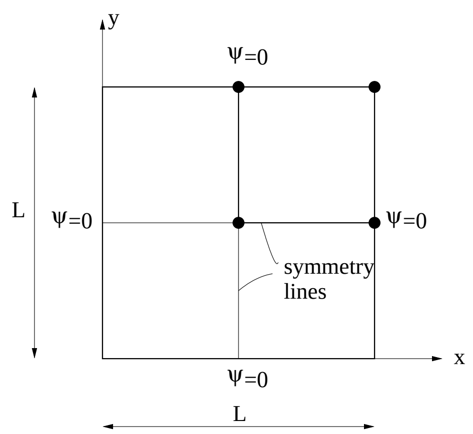
$$\int_A dA = \int_{-1}^{1}\int_{-1}^{1} det(\boldsymbol{J}) d\xi d\eta, \quad \boldsymbol{J} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{bmatrix}$$

Problem 5 : (12p)

The classical problem - a particle in a 2-dim box - is described by the time independent Schrödinger equation

$$\frac{\hbar}{2m}\mathrm{div}(\boldsymbol{\nabla}\psi) - E\psi = 0$$

where ψ is the wavefunction, $\hbar$ is the Planck-constant, m is the mass of the particle and E is the total (constant) energy in the system. The boundary conditions are given by



Determine the weak form as well as the FE-formulation associated with the problem. Use the symmetry shown in the figure and define the natural and essential boundary conditions.

Suggest a suitable (completeness and compatibility satisfied) approximation for the 4-node element shown in the figure above. Moreover, determine if any parasitic terms are present in the approximation.

Problem 6 : (10p)

In the final examination of the course 'Endimensionell analys 2', 2011-12-19 the students were asked to calculate the following integal

$$\int_{-2}^{-1} \frac{x+1}{x^2+4x+5}.$$

- Use three point Gauss's integration to evaluate this integral. Is the numerical integration result exact ? Under what circumstances does Gauss integration yield the exact result ?
- Explain the term reduced integration. Give two reasons why it is used.
- Explain the term spurious zero-energy modes.

Problem 7 (FHLF01) : (10p)

The problem defined by

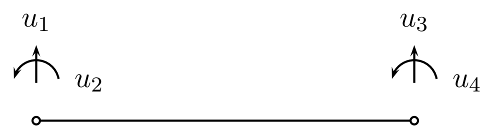
$$-div(\boldsymbol{\nabla} T) = Q \quad \text{over} \quad V$$

along with $T = 0$ over the boundary S is equivalent to the minimization problem

$$\begin{cases} J(T) = \int_V \frac{1}{2} (\boldsymbol{\nabla} T)^T \boldsymbol{\nabla} T dV - \int_V TQdV \\ T \in \{T \in C^2(V) |\ \text{T=0 on } S\} \end{cases}$$

Use the Ritz method to establish a linear system that governs the approximative solution. You may assume that the n basis functions φ_n are known.

Problem 7 (FHL064) : (10p)



For a beam of length a the interpolation for the displacement is given by

$$w = N_1^e u_1 + N_2^e u_2 + N_3^e u_3 + N_4^e u_4$$

a) Sketch the shape functions

b) Indicate in all sketches if the shape function or slope of the shape function has the value 1 at the nodal points.

Some hints that might be helpful

Hint: Green-Gauss's theorem states:

$$\int_A \phi \; div\boldsymbol{q} dA = \oint_{\mathcal{L}} \phi \boldsymbol{q}^T \boldsymbol{n} d\mathcal{L} - \int_A (\boldsymbol{\nabla}\phi)^T \boldsymbol{q} dA$$

Hint: The strain components are defined as

$$\epsilon_{xx} = \frac{\partial u_x}{\partial x}, \;\; \epsilon_{xx} = \frac{\partial u_y}{\partial y}, \;\; \epsilon_{zz} = \frac{\partial u_z}{\partial z}$$

and

$$\gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}, \;\; \gamma_{xz} = \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}, \;\; \gamma_{yz} = \frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z},$$

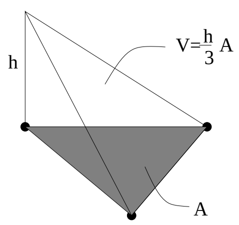
Hint: A quadratic matrix is positive semidefinite if

$$\boldsymbol{a}^T \boldsymbol{K} \boldsymbol{a} \geq 0, \quad \forall \boldsymbol{a}, \quad \text{and} \quad \boldsymbol{a}^T \boldsymbol{K} \boldsymbol{a} = 0 \quad \text{for some} \quad \boldsymbol{a} \neq \boldsymbol{0}$$

Hint: The interpolation formula of Lagrange is given by

$$l_k^{n-1} = \frac{(x - x_1).....(x - x_{k-1})(x - x_{k+1}).....(x - x_n)}{(x_k - x_1).....(x_k - x_{k-1})(x_k - x_{k+1}).....(x_k - x_n)}$$

Hint: Fourier's law is given by $\boldsymbol{q} = -\boldsymbol{D}\boldsymbol{\nabla} T$
Hint: Hooke's law for thermo-elasticity is given by $\boldsymbol{\sigma} = \boldsymbol{D}\left(\boldsymbol{\epsilon} - \boldsymbol{\epsilon}^o\right)$
Hint:



Hint: The position, ξ_i, of the integration points and weights, H_i, for n number of integration points can be found from

n	ξ_i	H_i
1	0	2
2	$-1/\sqrt{3}$, $1/\sqrt{3}$	1,1
3	0 , $\sqrt{3/5}$, $-\sqrt{3/5}$	8/9, 5/9 , 5/9

where integration from -1 to 1 is assumed.

Hint:

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & a & b \\ 1 & c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} (ad - bc) & 0 & 0 \\ -d + b & d & -b \\ c - a & -c & a \end{bmatrix}$$