Name: ______________________________ **NOTE: Hand in the exam!!!**

THE FINITE ELEMENT METHOD 2015
Dept. of Solid Mechanics

EXAMINATION: 2015-05-30

A maximum of 60 points can be obtained in this examination. To pass at least 30 points are required.

Permitted aid: Pocket calculator.

Problem 1: (12p)

The diffusion of Hydrogen molecules in a pressure vessel steel is modeled by the following parabolic partial differential equation

$$\frac{\partial c}{\partial t} = \text{div}(\boldsymbol{M}\boldsymbol{\nabla} c) + s$$

with c the concentration field of Hydrogen molecules. $\boldsymbol{M}$ is the mobility of the Hydrogen molecules in steel and s a source of Hydrogen molecules. Both, $\boldsymbol{M}$ and s are given.

The boundary of the analyzed domain, V, is divided into one part S_h where $(\boldsymbol{M}\boldsymbol{\nabla} c)^T\boldsymbol{n} = h$ with h a given function on S_h and one part S_c where $(\boldsymbol{M}\boldsymbol{\nabla} c)^T\boldsymbol{n} = (d + f\ c)$ with d and f experimentally determined constants.

a) Derive the weak form of the problem stated above.

b) Derive the finite element formulation corresponding to the problem above. Note that you do not need to calculate any matrices.

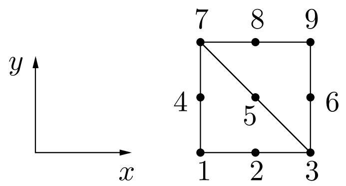
c) Suggest an element and show that the chosen element fulfills compatibility.

Problem 2: (12p)

The finite element formulation for two dimensional transient heat conduction takes the form

$$\Big(\underbrace{\int_A \boldsymbol{N}^T c\rho \boldsymbol{N} dA}_{\boldsymbol{C}}\Big)\dot{\boldsymbol{a}} + \Big(\underbrace{\int_A \boldsymbol{B}^T \boldsymbol{D}\boldsymbol{B} dA}_{\boldsymbol{K}} + \underbrace{\int_{\mathcal{L}_c} \alpha \boldsymbol{N}^T \boldsymbol{N} d\mathcal{L}}_{\tilde{\boldsymbol{K}}}\Big)\boldsymbol{a} = \underbrace{\int_{\mathcal{L}_c} \boldsymbol{N}^T \alpha T_\infty d\mathcal{L}}_{\boldsymbol{f}_c} \underbrace{- \int_{\mathcal{L}_g} \boldsymbol{N}^T q_n d\mathcal{L} - \int_{\mathcal{L}_h} \boldsymbol{N}^T h d\mathcal{L}}_{\boldsymbol{f}_b} + \underbrace{\int_A \boldsymbol{N}^T Q dA}_{\boldsymbol{f}_l}$$

where $\mathcal{L}_c$ is the boundary where convection applies, $\mathcal{L}_h$ is the boundary where the heat flow is prescribed and $\mathcal{L}_g$ is the boundary where the temperature is prescribed.



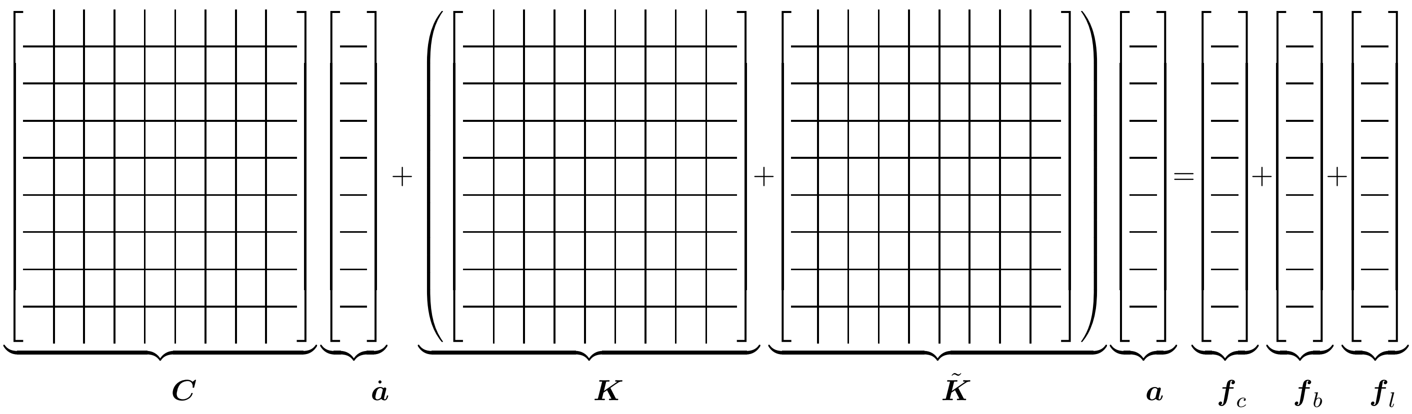
The boundary conditions for the problem meshed by two 6-node triangular elements are

- Prescribed temperature at nodes 1-2-3, i.e. $T = a =$ const.
- Prescribed heat flow trough the edge defined by the nodes 3-6-9, i.e. $q_n = h \neq 0$
- Convection at the edge defined by the nodes 1-4-7, i.e. $q_n = \alpha(T - T_\infty)$
- Insulated edge defined by the nodes 7-8-9, i.e. $q_n = 0$

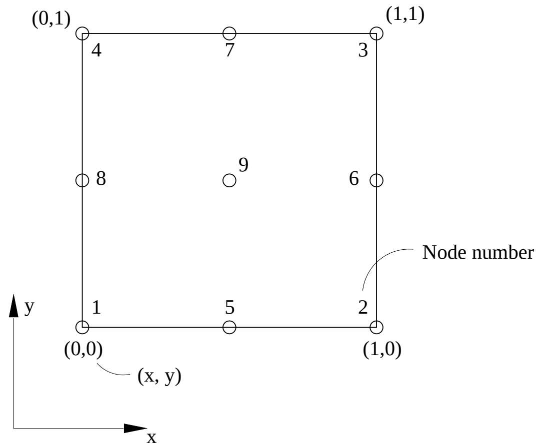
Additionally, heat is supplied via a constant heat source Q in the element defined by the nodes 3-6-9-8-7-5.

In the system of equations below mark

x – components that are known and different from zero
? – components that are unknown and different from zero
all blank positions are interpreted as zero.



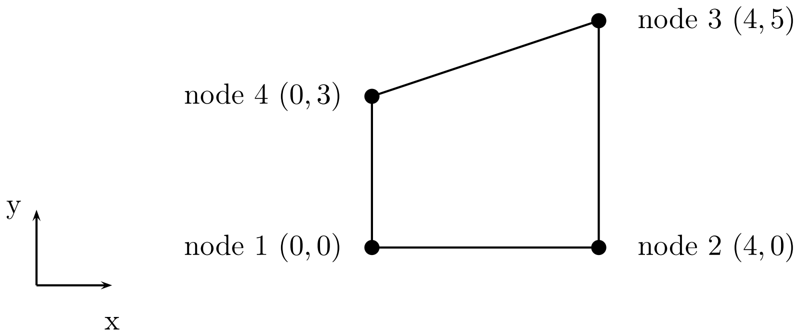
Problem 3: (12p)



A nine-node Lagrange element (dimensions in mm) has been used in a two-dimensional elastic FE-analysis. On the boundary 4-7-3 there is a constant external force acting between (0.25,1)-(0.75,1). The force is due to friction and pressure and is given by $F_x = 5N$ and $F_y = -25N$. Calculate the contribution from F_x and F_y to the boundary load vector defined as $\boldsymbol{f}_b = \int_{\mathcal{L}} \boldsymbol{N}^T \boldsymbol{t} t d\mathcal{L}$. The thickness t is $10mm$.

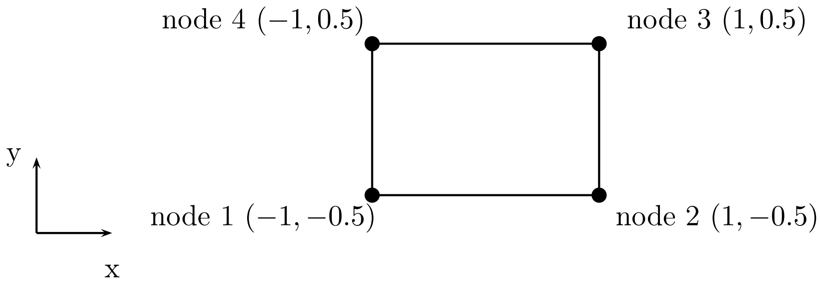
Problem 4: (12p)

The isoparametric fully integrated four node element depicted below will be used in a thermal analysis.



a) Determine the positions of the Gauss points, i.e. (x_i, y_i) for all four Gauss points.

b) Another element is located as in the figure below. Calculate the integral $\int_A x^2 y^2 dA$ using full integration, i.e. four integration points.



Problem 5 : (12p) FHLF10, Only Pi

The governing equations for a Bernoulli beam is given by

$$\frac{d^2M}{dx^2} + q = 0 \quad , \qquad M = -EI\frac{d^2w}{dx^2}$$

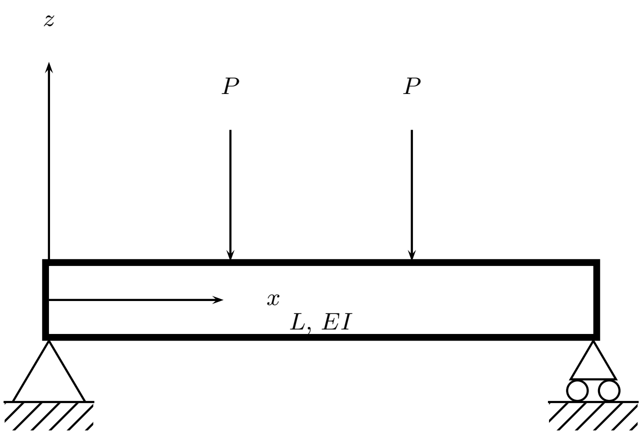
where q denotes the load intensity (positiv i z-direction). Note that the equilibrium equation is derived from the following relations

$$\frac{dM}{dx} = V \quad , \qquad \frac{dV}{dx} = -q$$

a) Derive the weak form of the governing equation and specify the essential and natural boundary conditions.
b) Derive the FE-formulation for the problem, such that a symmetric stiffness matrix is obtained.

c)

A beam is placed on two supports as shown in the figure below. The beam has the length L and bending stiffness EI. The beam is loaded with two forces of equal magnitude P at $x = L/3$ and $x = 2L/3$. Calculate the deflection at the center of the beam, i.e. at $x = L/2$.



Problem 5: (12p) FHLF01, Only F

The differential equation governing a column (simply supported in both ends) with an intial curvature is given by

$$\frac{d^2u}{dx^2} + \frac{1}{\pi^2}u + \sin x = 0 \qquad 0 \leq x \leq \pi$$
$$u(0) = u(\pi) = 0$$

Make use of the following approximation

$$\bar{u}(x) = \sum_{k=1}^{n} a_k \sin(kx)$$

1. Determine the residual $e(x)$.
2. Use the Galerkins method to determine the parameters a_k, $k = 1, 2, 3, , , , , n$.

Hint:

$$\int_0^{\pi} \sin(kx)\sin(lx)dx = \begin{cases} 0 & \text{om } k \neq l \\ \pi/2 & \text{om } k = l \end{cases}$$

Some hints that might be helpful

Hint: Green-Gauss's theorem states:

$$\int_A \phi div\boldsymbol{q} dA = \oint_{\mathcal{L}} \phi \boldsymbol{q}^T \boldsymbol{n} d\mathcal{L} - \int_A (\boldsymbol{\nabla}\phi)^T \boldsymbol{q} dA$$

Hint: Some trigonometric relations:

$$\sin(\alpha)^2 + \cos(\alpha)^2 = 1$$

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta), \quad \cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

$$\sin^2(\alpha) = \frac{1-\cos(2\alpha)}{2}, \quad \cos^2(\alpha) = \frac{1+\cos(2\alpha)}{2}$$

$$\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha), \quad \tan(2\alpha) = \frac{2\tan(\alpha)}{1-\tan^2(\alpha)}$$

Hint: A quadratic matrix is positive semidefinite if

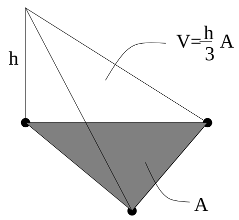
$$\boldsymbol{a}^T\boldsymbol{K}\boldsymbol{a} \geq 0, \quad \forall \boldsymbol{a}, \quad \text{and} \quad \boldsymbol{a}^T\boldsymbol{K}\boldsymbol{a} = 0 \quad \text{for some} \quad \boldsymbol{a} \neq \boldsymbol{0}$$

Hint: The interpolation formula of Lagrange is given by

$$l_k^{n-1} = \frac{(x-x_1).....(x-x_{k-1})(x-x_{k+1}).....(x-x_n)}{(x_k-x_1).....(x_k-x_{k-1})(x_k-x_{k+1}).....(x_k-x_n)}$$

Hint: Fourier's law is given by $\boldsymbol{q} = -\boldsymbol{D}\boldsymbol{\nabla}T$

Hint:



Hint: The position, ξ_i, of the integration points and weights, H_i, for n number of integration points can be found from

n	ξ_i	H_i
1	0	2
2	$\pm 1/\sqrt{3}$	1

where integration from -1 to 1 is assumed.

Hint:

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & a & b \\ 1 & c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} (ad-bc) & 0 & 0 \\ -d+b & d & -b \\ c-a & -c & a \end{bmatrix}$$