FINITE ELEMENT METHOD FHL064

SOLUTIONS TO THE FINALE EXAMINATION: 2011-06-03

Problem 1

The Laplace equation:

$$div(\boldsymbol{\nabla}\phi) = 0$$

a) The weak form:

$$\int_A (\boldsymbol{\nabla} v)^T \boldsymbol{\nabla}\phi dA - \oint_L v(\boldsymbol{\nabla}\phi)^T \boldsymbol{n} dL = 0$$

b) The FE-formulation:
Use the approximation: $\phi = \boldsymbol{N}\boldsymbol{a}$, $\boldsymbol{\nabla}\phi = \boldsymbol{B}\boldsymbol{a}$ with the Galerkin method: $v = \boldsymbol{N}\boldsymbol{c} = \boldsymbol{c}^T\boldsymbol{N}^T$, $\boldsymbol{\nabla} v = \boldsymbol{B}\boldsymbol{c} = \boldsymbol{c}^T\boldsymbol{B}^T$ where $\boldsymbol{c}$ is arbitrary.

$$\int_A \boldsymbol{B}^T\boldsymbol{B} dA\boldsymbol{a} - \oint_L \boldsymbol{N}^T(\boldsymbol{\nabla}\phi)^T \boldsymbol{n} dL = \boldsymbol{0}$$

c) The stiffness matrix

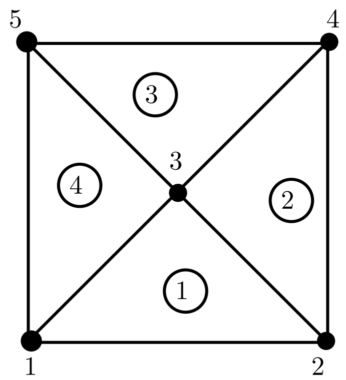

Figure 1: Mesh

Using the symmetry it is sufficient to calculate the stiffness matrix for one triangular element. To calculate the gradient of the shape functions $\boldsymbol{B}$ see the book at page: 118 - 125 . For element 1 we have:

$$\boldsymbol{B}^e = \begin{bmatrix} -1 & 1 & 0 \\ -1 & -1 & 2 \end{bmatrix}$$

which results in the stiffness matrix:

$$
\boldsymbol{K}^e = \int_A \boldsymbol{B}^{eT}\boldsymbol{B}^e dA = \boldsymbol{B}^{eT}\boldsymbol{B}^e A = 0.5 \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 2 \end{bmatrix}
$$

Using the edof-matrix:

$$
edof = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 2 & 2 & 4 & 3 \\ 3 & 4 & 5 & 3 \\ 4 & 5 & 1 & 3 \end{bmatrix}
$$

results in the global stiffness matrix:

$$
\boldsymbol{K} = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ -1 & -1 & 4 & -1 & -1 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{bmatrix}
$$

Problem 2

$$
\underbrace{\left(\left[\begin{array}{c|c|c|c|c|c|c|c|c}
x & x & x & x & & & & & \\ \hline
x & x & x & & x & & & & \\ \hline
x & x & x & x & x & & & & \\ \hline
x & & x & x & x & & x & x & \\ \hline
 & x & x & x & x & x & x & x & x \\ \hline
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 & & & & x & x & & x & x
\end{array}\right]\right.}_{\boldsymbol{K}}
+
\underbrace{\left.\left[\begin{array}{c|c|c|c|c|c|c|c|c}
x & & & x & & & & & \\ \hline
 & & & & & & & & \\ \hline
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x & & & x & & & x & & \\ \hline
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 & & & & & & & &
\end{array}\right]\right)}_{\tilde{\boldsymbol{K}}}
\underbrace{\begin{bmatrix} x \\ \hline x \\ \hline ? \\ \hline ? \\ \hline ? \\ \hline ? \\ \hline ? \\ \hline ? \\ \hline ? \end{bmatrix}}_{\boldsymbol{a}}
=
\underbrace{\begin{bmatrix} \\ \hline \\ \hline \\ \hline x \\ \hline x \\ \hline x \\ \hline \\ \hline x \\ \hline x \end{bmatrix}}_{\boldsymbol{f_l}}
+
\underbrace{\begin{bmatrix} ? \\ \hline ? \\ \hline \\ \hline \\ \hline x \\ \hline x \\ \hline \\ \hline \\ \hline x \end{bmatrix}}_{\boldsymbol{f_b}}
+
\underbrace{\begin{bmatrix} x \\ \hline \\ \hline \\ \hline x \\ \hline \\ \hline \\ \hline x \\ \hline \\ \hline \end{bmatrix}}_{\boldsymbol{f_c}}
$$

Problem 3

$$
\begin{bmatrix} f_{b1x} \\ f_{b1y} \end{bmatrix} = \int_0^{3a} \begin{bmatrix} N_1 & 0 \\ 0 & N_1 \end{bmatrix} \boldsymbol{t} dL
$$

Along the boundary, the shape function N_1 reduces to a cubic polynomial:

$$
N_1 = \frac{x(x-a)(x-2a)}{6a^3}
$$

$$
\begin{bmatrix} f_{b1x} \\ f_{b1y} \end{bmatrix} = \int_0^{a} \begin{bmatrix} N_1 & 0 \\ 0 & N_1 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{a} \end{bmatrix} dx + \int_0^{3a} \begin{bmatrix} N_1 & 0 \\ 0 & N_1 \end{bmatrix} \begin{bmatrix} \delta(2.5a) \\ \delta(2.5a) \end{bmatrix} dx
$$

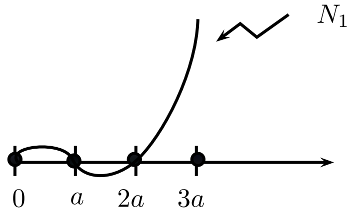

Figure 2: Shape function N_1 along the boundary

$$\begin{bmatrix} f_{b1x} \\ f_{b1y} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{a}\int_0^a \frac{x(x-a)(x-2a)}{6a^3}dx \end{bmatrix} + \begin{bmatrix} N_1(2.5a) \\ N_1(2.5a) \end{bmatrix}$$

$$\begin{bmatrix} f_{b1x} \\ f_{b1y} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{24} \end{bmatrix} + \begin{bmatrix} \frac{5}{16} \\ \frac{5}{16} \end{bmatrix} = \frac{1}{48}\begin{bmatrix} 15 \\ 17 \end{bmatrix}$$

Problem 4

For a two node element the gradient of the shape functions is given by:

$$\boldsymbol{B} = \frac{1}{L}\begin{bmatrix} -1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -1 & 1 \end{bmatrix}$$

then the stiffness matrix for the element is:

$$\boldsymbol{K} = \frac{1}{4}\begin{bmatrix} -1 \\ 1 \end{bmatrix}\begin{bmatrix} -1 & 1 \end{bmatrix}\int_2^4 k(x)dx = \frac{1}{4}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\int_2^4 k(x)dx$$

Use of the following mapping $\xi(x) = x - 3$ from the $x-$ to the $\xi-$ domain

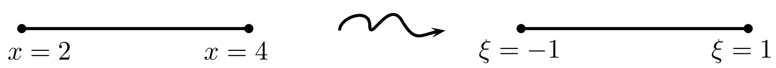

Figure 3: Illustration of mapping from $x-$ to $\xi-$ domain

results in

$$\int_2^4 k(x)dx = \int_{-1}^1 k(\xi)\frac{dx}{d\xi}d\xi = \int_{-1}^1 k(\xi)d\xi \approx \sum_{i=1}^{n=2} k(\xi_i)H_i$$

$$H_1 = 1, \quad \xi_1 = 0.577 \Rightarrow x_1 = 3.577 \Rightarrow k(3.577) \approx 3.2$$

$$H_2 = 1, \quad \xi_2 = -0.577 \Rightarrow x_2 = 2.423 \Rightarrow k(2.423) \approx 1.2$$

$$\boldsymbol{K} = 1.1\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Problem 5

a)

$$\frac{d^2M}{dx^2} + q = 0$$
$$\int v\frac{d^2M}{dx^2}dx + \int vqdx = 0$$
$$\int \frac{dv}{dx}\frac{dM}{dx}dx = [v\frac{dM}{dx}] + \int vqdx$$
$$\int \frac{d^2v}{dx^2}Mdx = [\frac{dv}{dx}M] - [v\frac{dM}{dx}] - \int vqdx$$

M, V are natural boundary conditions and w, $\frac{dw}{dx}$ are essential boundary conditions

b)

$$w = \boldsymbol{Na}, \quad \frac{d^2w}{dx^2} = \boldsymbol{Ba}, \quad v = \boldsymbol{Nc}, \quad \frac{d^2v}{dx^2} = \boldsymbol{Bc}$$
$$\int \boldsymbol{B}^T EI\boldsymbol{B}dx\boldsymbol{a} = [\boldsymbol{N}^T V] - [\frac{d\boldsymbol{N}^T}{dx}M] + \int \boldsymbol{N}^T qdx$$

Problem 6

$$\begin{aligned}\frac{d^2u}{dx^2} &+ u + x = 0 \\ &\Rightarrow \\ L(\bullet) &= \frac{d^2(\bullet)}{dx^2} + (\bullet) \\ g(x) &= x\end{aligned}$$

$$e = L(u^{app}) + g$$

$$u^{app} = \boldsymbol{\psi a} = \begin{bmatrix} \sin \pi x & x(x-1) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

Galerkin method:

$$V = \boldsymbol{\psi c}$$

$$\int_0^1 vedx = \boldsymbol{c}^T \int_0^1 \boldsymbol{\psi}^T edx = 0, \quad \boldsymbol{c} \text{ is arbitrary} \quad \Rightarrow \quad \int_0^1 \boldsymbol{\psi}^T edx = \mathbf{0}$$

$$\begin{bmatrix} \int_0^1 \psi_1 L(\psi_1)dx & \int_0^1 \psi_1 L(\psi_2)dx \\ \\ \int_0^2 \psi_2 L(\psi_1)dx & \int_0^1 \psi_2 L(\psi_2)dx \end{bmatrix} \begin{bmatrix} a_1 \\ \\ a_2 \end{bmatrix} = \begin{bmatrix} -\int_0^1 \psi_1 gdx \\ \\ -\int_0^1 \psi_2 gdx \end{bmatrix}$$

$$\begin{bmatrix} -4.4348 & 1.1442 \\ 1.1442 & -0.3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} -0.3183 \\ 0.0833 \end{bmatrix}$$

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0.0066 \\ -0.2525 \end{bmatrix}$$