THE FINITE ELEMENT METHOD 2013
Dept. of Solid Mechanics

Short solutions: 2013-06-05

Problem 1 :

a) Multiply

$$-\gamma\dot{\rho} = \rho + \gamma div(\boldsymbol{\nabla}\rho) - \lambda$$

by a scalar weight function, v, and integrate over the domain, V. Use Green-Gauss theorem on the divergence term. Split the boundary into $S = S_c + S_h$. The result is:

$$-\int_V v\gamma\dot{\rho}dV = \int_V v\rho dV - \int_V \gamma\,(\boldsymbol{\nabla}v)^T\,\boldsymbol{\nabla}\rho dV + \int_{S_c} v(1+\rho)dS - \int_V v\lambda dV$$

b) Use of the interpolation, $\rho = \boldsymbol{N}\boldsymbol{\rho}$ yields $\dot{\rho} = \boldsymbol{N}\dot{\boldsymbol{\rho}}$ and $\boldsymbol{\nabla}\rho = \boldsymbol{B}\boldsymbol{\rho}$. The weight function, v, is chosen as $v = \boldsymbol{N}\boldsymbol{c}$ which yields $\boldsymbol{\nabla}c = \boldsymbol{B}\boldsymbol{c}$. Insertion into the weak form results in

$$\int_V \gamma\boldsymbol{B}^T\boldsymbol{B}dV\boldsymbol{\rho} - \int_V \boldsymbol{N}^T\boldsymbol{N}dV\boldsymbol{\rho} - \int_V \boldsymbol{N}^T\gamma\boldsymbol{N}dV\dot{\boldsymbol{\rho}} - \int_{S_c}\boldsymbol{N}^T\boldsymbol{N}\boldsymbol{\rho}dS = \int_{S_c}\boldsymbol{N}^T dS - \int_V \boldsymbol{N}^T\lambda dV$$

Problem 2 :

The contribution to node 1 is given as

$$\{\boldsymbol{f}_b\}_1 = \int_{\mathcal{L}} N_1 \boldsymbol{t} d\mathcal{L} = \int_{\mathcal{L}_{right}} N \boldsymbol{t} d\mathcal{L} + \int_{\mathcal{L}_{left}} N_1 \boldsymbol{t} d\mathcal{L} = [symmetry] = 2 \int_{\mathcal{L}_{right}} N_1 \boldsymbol{t} d\mathcal{L}$$

In the element located to the right, the shape function associated with node 1 can be expressed as $N_1^e = (a-\eta)(2a-\eta)(3a-\eta)/(6a^3)$

$$2 \int_{\mathcal{L}_{right}} N_1 \boldsymbol{t} d\mathcal{L} = 2 \int_0^{3a} N_1 \boldsymbol{t} d\eta = 2 \int_{-1}^{1} N_1 \boldsymbol{t} \frac{d\eta}{d\xi} d\xi =$$

$$2 \left(\left(N_1 \boldsymbol{t} \frac{d\eta}{d\xi} \right)_{\xi=-0.57} + \left(N_1 \boldsymbol{t} \frac{d\eta}{d\xi} \right)_{\xi=0.57} \right) = 2 \left(N_1 \boldsymbol{t} \frac{d\eta}{d\xi} \right)_{\xi=-0.57}$$

Since $\xi = -1 + \frac{2\eta}{3a}$, we have that $\frac{d\eta}{d\xi} = \frac{3a}{2}$ and $\eta(\xi = -0.57) = \frac{3a}{2} 0.43$.

$$2 \left(N_1 \boldsymbol{t} \frac{d\eta}{d\xi} \right)_{\xi=-0.57} = 3a[b\ c]^T (a - \frac{3a}{2} 0.43)(2a - \frac{3a}{2} 0.43)(3a - \frac{3a}{2} 0.43)/(6a^3)$$

Problem 3

a) Premultiply and postmultiply $\boldsymbol{K} + \tilde{\boldsymbol{K}}$ by $\boldsymbol{a}^T$ and $\boldsymbol{a}$, respectively. Identifying $\boldsymbol{N}\boldsymbol{a} = T$ and $\boldsymbol{B}\boldsymbol{a} = \boldsymbol{\nabla} T$ reveals that $\boldsymbol{a}^T(\boldsymbol{K} + \tilde{\boldsymbol{K}})\boldsymbol{a} > 0$ for $\alpha > 0$.

b) An example of an approximation is $T = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 z + \alpha_5 xy + \alpha_6 xz + \alpha_7 yz + \alpha_8 xyz$. Show that the approximation satisfies the convergence criterion, i.e. completeness+compatibility requirements. See page 91-94.

Problem 4

The displacement, $\boldsymbol{u}$, is discretized as

$$\boldsymbol{u}(x,y) = \boldsymbol{N}(x,y)\boldsymbol{a} \tag{1}$$

where $\boldsymbol{N}$ and $\boldsymbol{a}$ are the shape functions and nodal values associated with the mechanical problem. The total strain is

$$\boldsymbol{\varepsilon} = \tilde{\boldsymbol{\nabla}}\boldsymbol{u} = \tilde{\boldsymbol{\nabla}}\boldsymbol{N}(x,y)\boldsymbol{a} = \boldsymbol{B}\boldsymbol{a} \quad \text{where} \quad \tilde{\boldsymbol{\nabla}} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \tag{2}$$

The stress is calculated using Hooke's law, i.e.

$$\boldsymbol{\sigma} = \boldsymbol{D}\boldsymbol{\varepsilon}^e = \boldsymbol{D}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^0) = \boldsymbol{D}(\boldsymbol{\varepsilon} - \alpha\Delta T(x,y)\begin{bmatrix}1 & 1 & 0\end{bmatrix}^T)$$

The temperature difference is $\Delta T = T(x,y) - T_0$, where T_0 is the stress free temperature and the current temperature, T, is discretized as

$$T(x,y) = \boldsymbol{N}^0\boldsymbol{a}^0$$

where $\boldsymbol{N}^0$ and $\boldsymbol{a}^0$ are the shape functions and nodal values associated with the temperature problem.

Problem 5

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\boldsymbol{K} = \begin{bmatrix}
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\end{bmatrix}
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\tilde{\boldsymbol{K}} = \begin{bmatrix}
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 & & & & & & & & & & & & x & x & x
\end{bmatrix}
$$

$$
\boldsymbol{a} = \begin{bmatrix} x \\ x \\ x \\ x \\ x \\ ? \\ ? \\ ? \\ ? \\ ? \\ ? \\ ? \\ ? \\ ? \\ ? \end{bmatrix}
\quad
\boldsymbol{f}_l = \begin{bmatrix} x \\ x \\ x \\ \\ \\ x \\ x \\ x \\ \\ \\ x \\ x \\ x \\ \\ \\ \end{bmatrix}
\quad
\boldsymbol{f}_b = \begin{bmatrix} ? \\ ? \\ ? \\ ? \\ ? \\ x \\ \\ \\ \\ x \\ x \\ \\ \\ \\ x \end{bmatrix}
\quad
\boldsymbol{f}_c = \begin{bmatrix} \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ x \\ x \\ x \\ x \\ x \end{bmatrix}
$$

Problem 6 FHLF01

a)
One possible ansatz with the unknowns a_1 and a_2 which fulfils the boundary conditions is,

$$\psi = \big(\sin(\pi x)\sin(\pi y)\big)a_1 + \big(\sin(2\pi x)\sin(2\pi y)\big)a_2$$

This can also be written as

$$\psi = [\sin(\pi x)\sin(\pi y), \sin(2\pi x)\sin(2\pi y)] \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = [N_1, N_2] \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \boldsymbol{N}\boldsymbol{a}$$

b)

Define

$$\boldsymbol{B} = \nabla \boldsymbol{N} = \begin{bmatrix} \frac{dN_1}{dx}, & \frac{dN_2}{dx} \\ \\ \frac{dN_1}{dy}, & \frac{dN_2}{dy} \end{bmatrix}$$

Stationarity of the functional then provides

$$\frac{d}{d\epsilon}\Pi(\boldsymbol{a}+\epsilon\boldsymbol{c})\big|_{\epsilon=0} =$$

$$\tfrac{d}{d\epsilon}\left(\int_\Omega \tfrac{1}{2}(\boldsymbol{a}+\epsilon\boldsymbol{c})^T\boldsymbol{B}^T\boldsymbol{B})(\boldsymbol{a}+\epsilon\boldsymbol{c}) - (\boldsymbol{a}+\epsilon\boldsymbol{c})^T\boldsymbol{N}^T[2(x+y)-4]dV\right)\big|_{\epsilon=0} =$$

$$\boldsymbol{c}^T\left(\left(\int_\Omega \boldsymbol{B}^T\boldsymbol{B}dV\right)\boldsymbol{a} - \int_\Omega \boldsymbol{N}^T[2(x+y)-4]dV\right)$$

Since this expression should be true for arbitrary variations $\boldsymbol{c}$, then the following equation holds,

$$\boldsymbol{K}\boldsymbol{a} = \boldsymbol{f}$$

where

$$\boldsymbol{K} = \int_\Omega \boldsymbol{B}^T\boldsymbol{B}dV \quad \boldsymbol{f} = \int_\Omega \boldsymbol{N}^T[2(x+y)-4]dV$$

Problem 6 (FHL064) : (10p)

See Chapter 17 in Ottosen.