Problem 1

a) Inserting $u_1 = 1$, $F_2 = 2$ and $F_3 = 3$ yields

$$\begin{bmatrix} 12 & -4 & -8 \\ -4 & 6 & -2 \\ -8 & -2 & 10 \end{bmatrix} \begin{bmatrix} 1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} F_1 \\ 2 \\ 3 \end{bmatrix}$$

The last two rows gives

$$\begin{bmatrix} 6 & -2 \\ -2 & 10 \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 11 \end{bmatrix}$$

$$\begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 41/28 \\ 39/28 \end{bmatrix}$$

Using this in the first row yields $F_1 = -1$

b) If no displacement boundary conditions are introduced, there is nothing that prevents rigid body motion. This means that there are infinitely many solutions as the system may be located anywhere in space.

Problem 2

Use

$$\sum_i \mathbf{N}_i = 1 \qquad \int_V div\mathbf{q} \, \mathrm{d}V = \int_S q_n \, \mathrm{d}S$$

then

$$\sum_i \mathbf{f}_i = \int_V \sum \mathbf{N}^T Q \, \mathrm{d}V - \int_S \sum \mathbf{N}^T q_n \, \mathrm{d}S = \int_V Q \, \mathrm{d}V - \int_S q_n \, \mathrm{d}S = 0$$

Problem 3

Node 3 gets contribution from the adjacent elements (here termed el. 2 and 3) sharing the node. The contribution from one element

$$\mathbf{f}_b^{\mathbf{e}} = \int_{\mathcal{L}} t \begin{bmatrix} N_i & 0 \\ 0 & N_i \\ N_j & 0 \\ 0 & N_j \\ N_k & 0 \\ 0 & N_k \end{bmatrix} \begin{bmatrix} t_x \\ t_y \end{bmatrix} \mathrm{d}\mathcal{L} = \int_{\mathcal{L}} t \begin{bmatrix} N_i t_x \\ N_i t_y \\ N_j t_x \\ N_j t_y \\ N_k t_x \\ N_k t_y \end{bmatrix} \mathrm{d}\mathcal{L} = \begin{bmatrix} f_{b,ix} \\ f_{b,iy} \\ f_{b,jx} \\ f_{b,jy} \\ f_{b,kx} \\ f_{b,ky} \end{bmatrix}$$

If the element above node 3 is called element 2 and the element below node 3 called element 3 the total contribution to node 3 in the x-direction is

$$f_{b,3x} = \int_L t N_k^{el.2}(\xi) t_x^{el.2}(\xi) \, \mathrm{d}\xi + \int_L t N_i^{el.3}(\xi) t_x^{el.3}(\xi) \, \mathrm{d}\xi$$

where t_x are x-components of the traction vectors to the respective element and ξ is a local coordinate system introduced along the element boundaries. The traction vector in this particular case can be written as

$$\mathbf{t} = -p\mathbf{n} = -\rho g h(\xi) \begin{bmatrix} n_x \\ n_y \end{bmatrix}$$

where $\mathbf{n}$ is the normal to the element boundary and $h(\xi)$ is the depth as a function of the introduced variable ξ defined with help of the shape functions as

$$h(\xi) = h_i N_i(\xi) + h_j N_j(\xi)$$

where h_i is the depth at node i. The shape functions $N_{i,j,k}$ along the boundary for the two elements, with the local ξ-axis introduced, are defined as

$$N_i^{el.2} = 1 - \frac{\xi}{L_2}, \qquad N_k^{el.2} = \frac{\xi}{L_2}$$
$$N_i^{el.3} = 1 - \frac{\xi}{L_3}, \qquad N_k^{el.3} = \frac{\xi}{L_3}$$

The lengths of the element boundaries; L_2 and L_3 defines the base of a isosceles triangle meaning that they are given by

$$L_2 = L_3 = L = 2r \sin\left(\frac{20^\circ}{2}\right) \approx 5.2$$

Specific for the two elements at hand

$$h(\xi)^{el.2} = h_2 N_i^{el.2}(\xi) + h_3 N_k^{el.2}(\xi) = h_2 \left[1 - \frac{\xi}{L}\right] + h_3 \left[\frac{\xi}{L}\right] = h_2 + \frac{\xi}{L}\left[h_3 - h_2\right]$$
$$h(\xi)^{el.3} = h_3 N_i^{el.3}(\xi) + h_4 N_k^{el.3}(\xi) = h_3 \left[1 - \frac{\xi}{L}\right] + h_4 \left[\frac{\xi}{L}\right] = h_3 + \frac{\xi}{L}\left[h_4 - h_3\right]$$

where the specific values for the depths are given by

node	depth
h_2	$\sin(20^\circ)r - 1 \approx 4.13$
h_3	$\sin(40^\circ)r - 1 \approx 8.61$
h_4	$\sin(60^\circ)r - 1 \approx 12.00$

The normal vectors are element wise constant and given by

$$\mathbf{n}^{el.2} = \begin{bmatrix}\cos 30^\circ \\ \sin 30^\circ\end{bmatrix}, \qquad \mathbf{n}^{el.3} = \begin{bmatrix}\cos 50^\circ \\ \sin 50^\circ\end{bmatrix}$$

Finally we have

$$\begin{aligned}
f_{b,3x} &= t\int_L N_k^{el.2} t_x^{el.2} \mathrm{d}\xi + t\int_L N_i^{el.3} t_x^{el.3} \mathrm{d}\xi = \\
&= -t\rho g \int_L \frac{\xi}{L}\left[h_2 + \frac{\xi}{L}\left[h_3 - h_2\right]\right] \mathrm{d}\xi \cos(30^\circ) + \\
&\quad - t\rho g \int_L \left[1 - \frac{\xi}{L}\right]\left[h_3 + \frac{\xi}{L}\left[h_4 - h_3\right]\right] \mathrm{d}\xi \cos(50^\circ) = \\
&= -t\rho g \left[\frac{\xi^2}{2L} h_2 + \frac{\xi^3}{3L^2}\left[h_3 - h_2\right]\right]_0^L \cos(30^\circ) + \\
&\quad - t\rho g \left[\xi h_3 - \frac{\xi^2}{2L} h_3 + \frac{\xi^2}{2L}\left[h_4 - h_3\right] - \frac{\xi^3}{3L^2}\left[h_4 - h_3\right]\right]_0^L \cos(50^\circ) = \\
&= -t\rho g \left[\frac{L}{6} h_2 + \frac{L}{3} h_3\right] \cos(30^\circ) - t\rho g \left[\frac{L}{3} h_3 + \frac{L}{6} h_4\right] \cos(50^\circ) = \\
&= -t\rho g \frac{L}{6}\left[h_2 \cos(30^\circ) + 2h_3\left[\cos(30^\circ) + \cos(50^\circ)\right] + h_4 \cos(50^\circ)\right] \approx 32.44
\end{aligned}$$

Problem 4

Interpolation in isoparametric elements:

$$x(\xi,\eta) = \mathbf{N}^e(\xi,\eta)\mathbf{x}^e, \qquad y(\xi,\eta) = \mathbf{N}^e(\xi,\eta)\mathbf{y}^e$$

Stiffness matrix:

$$\mathbf{B}^e = \begin{bmatrix} \frac{\partial \mathbf{N}^e}{\partial x} \\ \frac{\partial \mathbf{N}^e}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} \end{bmatrix} \begin{bmatrix} \frac{\partial \mathbf{N}^e}{\partial \xi} \\ \frac{\partial \mathbf{N}^e}{\partial \eta} \end{bmatrix} = \mathbf{J}^{-T} \begin{bmatrix} \frac{\partial \mathbf{N}^e}{\partial \xi} \\ \frac{\partial \mathbf{N}^e}{\partial \eta} \end{bmatrix}$$

$$\mathrm{d}x\,\mathrm{d}y = \det \mathbf{J} \mathrm{d}\xi\,\mathrm{d}\eta$$

$$\mathbf{K}^e = \int \mathbf{B}^T(x,y)\mathbf{D}\mathbf{B}(x,y)\,\mathrm{d}x\,\mathrm{d}y =$$

$$= k\int_{-1}^{1}\int_{-1}^{1} \begin{bmatrix} \frac{\partial \mathbf{N}^e(\xi,\eta)}{\partial \xi} & \frac{\partial \mathbf{N}^e(\xi,\eta)}{\partial \eta} \end{bmatrix} \mathbf{J}^{-1}\mathbf{J}^{-T} \begin{bmatrix} \frac{\partial \mathbf{N}^e(\xi,\eta)}{\partial \xi} \\ \frac{\partial \mathbf{N}^e(\xi,\eta)}{\partial \eta} \end{bmatrix} \det \mathbf{J}\,\mathrm{d}\xi\,\mathrm{d}\eta$$

Problem 5

Multiply with arbitrary weight function and integrate over the body.

$$\int_A tv\frac{h}{2m}div(\boldsymbol{\nabla}\psi)\,\mathrm{d}A - \int_A tvE\psi\,\mathrm{d}A = 0$$

Use Green-gauss theorem

$$\int_A tv\frac{h}{2m}div(\boldsymbol{\nabla}\psi)\,\mathrm{d}A = \oint_{\mathcal{L}} vt\frac{h}{2m}(\boldsymbol{\nabla}\psi)^T\mathbf{n}\,\mathrm{d}\mathcal{L} - \int_A tv\frac{h}{2m}(\boldsymbol{\nabla}v)^T(\boldsymbol{\nabla}\psi)\,\mathrm{d}A$$

Insert and rearrange, assume constant thickness.

$$\int_A \frac{h}{2m}(\boldsymbol{\nabla}v)^T(\boldsymbol{\nabla}\psi)\,\mathrm{d}A = \oint_{\mathcal{L}} v\frac{h}{2m}(\boldsymbol{\nabla}\psi)^T\mathbf{n}\,\mathrm{d}\mathcal{L} - \int_A vE\psi\,\mathrm{d}A$$

The boundary integral is divided into essential and natural boundary conditions.

$$\int_A \frac{h}{2m}(\boldsymbol{\nabla}v)^T(\boldsymbol{\nabla}\psi)\,\mathrm{d}A = \oint_{\mathcal{L}_1} v\frac{h}{2m}(\boldsymbol{\nabla}\psi)^T\mathbf{n}\,\mathrm{d}\mathcal{L}_1 + \oint_{\mathcal{L}_2} v\frac{h}{2m}(\boldsymbol{\nabla}\psi)^T\mathbf{n}\,\mathrm{d}\mathcal{L}_2 - \int_A vE\psi\,\mathrm{d}A$$

where $\boldsymbol{\nabla}\psi$ is known at $\mathcal{L}_1$ (natural) and ψ is known on $\mathcal{L}_2$ (essential). For the FE formulation, choose v according to Galerkin and insert the approximation for the unknown ψ.

$$v = \mathbf{Nc} \qquad \boldsymbol{\nabla}v = \mathbf{Bc}$$
$$\psi = \mathbf{Na}, \qquad \boldsymbol{\nabla}\psi = \mathbf{Ba}$$

Insert into the weak formulation and using that $\mathbf{c}$ is arbitrary reults in

$$\int_A \mathbf{N}^T\mathbf{N}\psi\,\mathrm{d}A\mathbf{a} + \int_A \mathbf{B}^T\frac{h}{2m}\mathbf{B}\,\mathrm{d}A\,\mathbf{a} = \oint_{\mathcal{L}_1} \mathbf{N}^T\frac{h}{2m}(\boldsymbol{\nabla}\psi)^T\mathbf{n}\,\mathrm{d}\mathcal{L}_1 + \oint_{\mathcal{L}_2} \mathbf{N}^T\frac{h}{2m}(\boldsymbol{\nabla}\psi)^T\mathbf{n}\,\mathrm{d}\mathcal{L}_2$$

Using the symmetry, one element is sufficient to solve the problem. Along the symmetry lines ($\mathcal{L}_1$), $(\boldsymbol{\nabla}\psi)^T\mathbf{n} = 0$. Along ($\mathcal{L}_2$), $\psi = 0$. Suitable approximation for the 4-node element

$$\psi = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy$$

The approximation is able to describe both an arbitrary constant value and an arbitrary constant gradient, i.e. completeness is fulfilled. Along boundary where x is constant the approximation reduces down to

$$\psi = \beta_1 + \beta_2 y$$

Along boundary where y is constant the approximation reduces down to

$$\psi = \gamma_1 + \gamma_2 x$$

With two nodes at each boundary the approximation is compatible. One parasitic term is present; $\alpha_4 xy$.

Problem 6

$$\int_{-2}^{-1} \frac{x+1}{x^2+4x+5} dx$$

The integration interval can be transformed to [-1,1] by using

$$\begin{bmatrix} \xi = 2x+3 \\ x = 0.5\xi - 1.5 \\ dx = 0.5 d\xi \end{bmatrix}$$

This gives the integral

$$I = \int_{-1}^{1} \frac{0.5\xi - 1.5 + 1}{(0.5\xi - 1.5)^2 + 4(0.5\xi - 1.5) + 5} 0.5 d\xi = \int_{-1}^{1} \frac{\xi - 1}{\xi^2 + 2\xi + 5} d\xi = \int_{-1}^{1} f(\xi) d\xi$$

Using

$$\int_{-1}^{1} f(\xi) d\xi \approx \sum_i f(\xi_i) H_i$$

and three integration points give

$$I = \frac{5}{9} f(-\sqrt{3/5}) + \frac{8}{9} f(0) + \frac{5}{9} f(\sqrt{3/5}) \approx -0.438674$$

The numerical integration result is not exact, since the integrand is not a polynomial. Gauss integration with n integration points yields exact results for polynomials of order $\leq 2n-1$.

Reduced integration means that the order of integration is lower than required for obtaining an exact result. One reason to use reduced integration is that the computational cost becomes lower. Another reason is that reduced integration might increase the accuracy of the FE solution since it tends to soften the stiffness.

Spurious zero energy modes are displacement modes, other than those corresponding to rigid body motions, which creates zero strain energy.

Problem 7 FHL064

See course book page 326, Figure 17.8

Problem 7 FHLF01

See kontinuerliga system, pp240-241