Finite element formulation of transient heat transfer

For a material where the specific heat c and density ρ are constants the general form for the heat transfer equation reads

$$c\rho\dot{T} + div(\boldsymbol{q}) - Q = 0 \tag{1}$$

Multiplication by an arbitrary (time-independent) weight function and making use of the Green-Gauss theorem results in

$$\int_S v\boldsymbol{q}^T\boldsymbol{n}dS - \int_V (\boldsymbol{\nabla}v)^T\boldsymbol{q}dV - \int_V vQdV + \int_V vc\rho\dot{T}dV = 0 \tag{2}$$

The weight function is chosen as $v(x) = \boldsymbol{N}(\boldsymbol{x})\boldsymbol{c}$, i.e. the Galerkin approximation is adopted. Insertion of the weight function, v, in to (2) results in

$$\boldsymbol{c}^T\left\{\int_S \boldsymbol{N}^T\boldsymbol{q}^T\boldsymbol{n}dS - \int_V \boldsymbol{B}^T\boldsymbol{q}dV - \int_V \boldsymbol{N}^TQdV + \int_V \boldsymbol{N}^Tc\rho\dot{T}dV\right\} = 0 \tag{3}$$

Since $\boldsymbol{c}$ can be chosen arbitrarily we obtain

$$\int_S \boldsymbol{N}^T\boldsymbol{q}^T\boldsymbol{n}dS - \int_V \boldsymbol{B}^T\boldsymbol{q}dV - \int_V \boldsymbol{N}^TQdV + \int_V \boldsymbol{N}^Tc\rho\dot{T}dV = 0 \tag{4}$$

The spatial interpolation is chosen as

$$T = \boldsymbol{N}\boldsymbol{a} \tag{5}$$

It should be noted that the spatial dependence is captured in $\boldsymbol{N} = \boldsymbol{N}(\boldsymbol{x})$ whereas the time dependence is included in the nodal temperatures, i.e. $\boldsymbol{a} = \boldsymbol{a}(t)$. Insertion of (5) into (6) results in

$$\int_S \boldsymbol{N}^Tq_ndS + \int_V \boldsymbol{B}^T\boldsymbol{D}\boldsymbol{B}dV\boldsymbol{a} - \int_V \boldsymbol{N}^TQdV + \int_V \boldsymbol{N}^Tc\rho\boldsymbol{N}dV\dot{\boldsymbol{a}} = 0 \tag{6}$$

where use were made of the Fourier law, i.e. $\boldsymbol{q} = -\boldsymbol{D}\boldsymbol{\nabla}T$. Rearranging (6) results in

$$\boldsymbol{C}\dot{\boldsymbol{a}} + \boldsymbol{K}\boldsymbol{a} = \boldsymbol{f}_b + \boldsymbol{f}_l \tag{7}$$

where

$$\boldsymbol{C} = \int_V \boldsymbol{N}^Tc\rho\boldsymbol{N}dV, \quad \boldsymbol{K} = \int_V \boldsymbol{B}^T\boldsymbol{D}\boldsymbol{B}dV, \quad \boldsymbol{f}_l = \int_V \boldsymbol{N}^TQdV, \quad \boldsymbol{f}_b = -\int_S \boldsymbol{N}^Tq_ndS \tag{8}$$

Time integration procedures for transient heat transfer

To derive a relatively simple family of time integration schemes, let us start to make a an approximation for the time derivative in (7). The most simple way to estimate the derivative is to make use of two time steps that are close in time, i.e.

$$\dot{\boldsymbol{a}} = \frac{\boldsymbol{a}(t_{n+1}) - \boldsymbol{a}(t_n)}{t_{n+1} - t_n} \tag{9}$$

where t_{n+1} and t_n represent the time at two instants. Next, we will make use of the following approximations for $\boldsymbol{a}$ and $\boldsymbol{f}$

$$\begin{aligned} \boldsymbol{a} &= \theta \boldsymbol{a}_{n+1} + (1-\theta)\boldsymbol{a}_n, \\ \boldsymbol{f} &= \theta \boldsymbol{f}_{n+1} + (1-\theta)\boldsymbol{f}_n, \end{aligned} \tag{10}$$

where the notation $[\cdot]_{n+1} = [\cdot](t_{n+1})$ were for simplicity introduced. Note that θ is a numerical parameter that has to be chosen. Insertion of (9) and (10) into (7) result in

$$\boldsymbol{C}\frac{\boldsymbol{a}(t_{n+1}) - \boldsymbol{a}(t_n)}{t_{n+1} - t_n} + \boldsymbol{K}(\theta \boldsymbol{a}_{n+1} + (1-\theta)\boldsymbol{a}_n) = \theta \boldsymbol{f}_{n+1} + (1-\theta)\boldsymbol{f}_n \tag{11}$$

No definite answer regarding the choice of θ can be made, but the most common schemes can be summarized in

- $\theta = 1$. This is known as an implicit scheme and is often chosen due to it's stability properties
- $\theta = 1/2$. Midpoint rule or Crank-Nicholson scheme. Often used due to its accuracy properties.
- $\theta = 0$. Forward scheme or explicit scheme. If additional assumption regarding the $\boldsymbol{C}$ matrix is made, this method enables a very efficient scheme on large scale clusters to be obtained.