Some examples

Principle of minimum energy - Discrete system systems

We have previously derived the equilibrium equations $\boldsymbol{K}\boldsymbol{a} = \boldsymbol{f}$ for a system of springs based on equilibrium of the nodes. An alternative approach to derive the equilibrium equations can be found by considering the potential Π defined as

$$\Pi(\boldsymbol{a}) = W(\boldsymbol{a}) - \boldsymbol{a}^T\boldsymbol{F} \tag{1}$$

where $W = \frac{1}{2}\boldsymbol{a}^T\boldsymbol{K}\boldsymbol{a}$ is the stored energy of the system and $\boldsymbol{a}^T\boldsymbol{F}$ is referred to as the potential due to the external load. Note that in the definition of Π the external force $\boldsymbol{F}$ is constant. Minimization of the potential Π implies that Π should be stationary and therefore

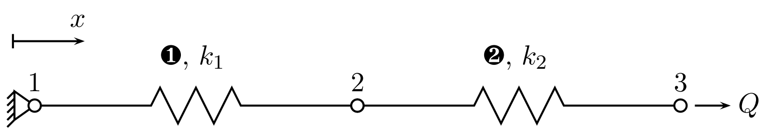
$$\frac{\partial \Pi}{\partial \{\boldsymbol{a}\}_i} = 0 \tag{2}$$

must hold, where $\{\boldsymbol{a}\}_i$ represents the i:th component of $\boldsymbol{a}$. Due to symmetry of the stiffness matrix this expression can be rewritten as

$$\frac{\partial \Pi}{\partial \{\boldsymbol{a}\}_i} = \sum_{j=1}^{ndof} \boldsymbol{K}_{ij}\boldsymbol{a}_j - \{\boldsymbol{F}\}_i = 0 \quad \forall i \tag{3}$$

i.e. we have found that a minimization of the potential Π implies the equilibrium. In many textbook the principle of minimum energy is taken as the basis for the finite element formulation.

Let us now turn to the 'two-spring' example shown in Fig. 1. The total



Figur 1: Illustration of a two connected springs loaded in tension.

stored energy for this system is given as the sum of the stored energy in each spring. For spring 1 the elongation, Δ_1, is equal to u_2 since $u_1 = 0$. For spring 2 the elongation, Δ_2, is $\Delta_2 = u_3 - u_2$. The total stored energy can now be expressed as

$$W(u_2, u_3) = \frac{k_1 u_2^2}{2} + \frac{k_2(u_3 - u_2)^2}{2} \tag{4}$$

Referring to (1) the total potential, Π, for the system can be written as

$$\Pi = W - Qu_3 = \frac{k_1 u_2^2}{2} + \frac{k_2(u_3 - u_2)^2}{2} - Qu_3 \tag{5}$$

The minumum of Π is found by requiring $\frac{\partial \Pi}{\partial u_2} = 0$ and $\frac{\partial \Pi}{\partial u_3} = 0$, i.e.

$$\begin{aligned} \frac{\partial \Pi}{\partial u_3} &= k_1 u_2 - k_2(u_3 - u_2) = 0 \\ \frac{\partial \Pi}{\partial u_3} &= k_2(u_3 - u_2) - Q = 0 \end{aligned} \tag{6}$$

These two equations can be written in matrix format as

$$\begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} \begin{bmatrix} 0 \\ Q \end{bmatrix} \tag{7}$$

i.e. the minimium potential energy enable us to form $\boldsymbol{K}\boldsymbol{a} = \boldsymbol{f}$.

Principle of minimum energy - continous system

For an axially loaded bar the potential energy can be expressed as

$$\Pi = \underbrace{\int_0^L \frac{1}{2} AE\varepsilon^2 dx}_{W} - \int_0^L bu dx - [uN]_0^L \tag{8}$$

where $\varepsilon = \frac{du}{dx}$. We shall now prove that a minimum to Π corresponds is an equilibruim solution. For this reason we assume that u is an equilibrium solution. If u minimizies the potential Π then $\Pi(u) \leq \Pi(u^*)$ or $\Pi(u) - \Pi(u^*) \leq 0$ for all choices of u^*. If we chose $u^* = u + v$ where v is a function that satisfies the essential boundary conditions if follows that u^* satisfies the essential boundary conditions. Using the definition for Π we obtain

$$\begin{aligned} \Delta\Pi(u, u^*) = \Pi(u) - \Pi(u^*) = \int_0^L \left(\frac{1}{2} AE \left(\frac{du}{dx} \right)^2 - \frac{1}{2} AE \left(\frac{du}{dx} + \frac{dv}{dx} \right)^2 \right) dx \\ - \int_0^L bu dx - [uN]_0^L + \int_0^L b(u + v) dx + [(u + v)N]_0^L \end{aligned} \tag{9}$$

Expansion and simplification of (9) results in

$$\Delta\Pi(u, u^*) = - \int_0^L \left(AE \left(\frac{dv}{dx} \frac{du}{dx} + \frac{1}{2} \left(\frac{dv}{dx} \right)^2 \right) \right) dx + \int_0^L bv dx + [vN]_0^L \tag{10}$$

Since u is an equilibrium solution it must satisfy the weak form. Using this result we conclude that

$$\Delta\Pi(u, u^*) = \Pi(u) - \Pi(u^*) = - \int_0^L AE \frac{1}{2} \left(\frac{dv}{dx} \right)^2 dx \leq 0 \tag{11}$$

and we conclude that $\Pi(u) \leq \Pi(u^*)$, i.e. the displacement field that is minimizing the potential Π is soving the equlibrium.

Principle of minimum energy - General elasticity

Assume that a strain energy potential exsists, i.e. $w = w(\boldsymbol{\varepsilon})$ where $\boldsymbol{\varepsilon} = \tilde{\boldsymbol{\nabla}}\boldsymbol{u}$. An obvious generalization of (8) reads

$$\Pi(\boldsymbol{u}^*) = \underbrace{\int_\Omega w dV}_{W} - \int_{\partial\Omega_t} \boldsymbol{t}^T \boldsymbol{u}^* dS \tag{12}$$

where we require that $\boldsymbol{u}$ safisfies the essential boundary conditions. Suppose that $\boldsymbol{u}$ is a displacement field that satisfies equilibrium. In that case Π takes a minimal value for $\Pi(\boldsymbol{u})$. The minimization principle may be reformulated as

$$\Pi(\boldsymbol{u}) \leq \Pi(\boldsymbol{u}^*), \quad \forall \boldsymbol{u}^* \tag{13}$$

Let us now define $\boldsymbol{u} = t\boldsymbol{v}$ where t is a scalar. Using this definition we can reformulate the minimization problem (13) as

$$\frac{d\Pi\left(\boldsymbol{u} + t\boldsymbol{v}\right)}{dt}|_{t=0} = 0, \quad \forall \boldsymbol{v} \tag{14}$$

Let is now assume that the material is linear elastic, i.e.

$$w = \frac{1}{2}\boldsymbol{\varepsilon}^T \boldsymbol{D} \boldsymbol{\varepsilon} \tag{15}$$

where $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\boldsymbol{u})$. Using (12), (14) and (15) we obtain

$$\begin{aligned} &\frac{d\Pi\left(\boldsymbol{u} + t\boldsymbol{v}\right)}{dt}|_{t=0} = 0 = \\ &\frac{d}{dt}|_{t=0} \left\{ \int_\Omega \frac{1}{2} \left(\tilde{\boldsymbol{\nabla}}(\boldsymbol{u} + t\boldsymbol{v})\right)^T \boldsymbol{D} \left(\tilde{\boldsymbol{\nabla}}(\boldsymbol{u} + t\boldsymbol{v})\right) dV - \int_{\partial\Omega_t} \boldsymbol{t}^T (\boldsymbol{u} + t\boldsymbol{v}) ds \right\} \end{aligned} \tag{16}$$

After expaning the terms in (16) and using $\boldsymbol{D} = \boldsymbol{D}^T$ we obtain

$$\int_\Omega \left(\tilde{\boldsymbol{\nabla}}\boldsymbol{v}\right)^T \boldsymbol{D} \left(\tilde{\boldsymbol{\nabla}}\boldsymbol{u}\right) dV - \int_{\partial\Omega_t} \boldsymbol{t}^T \boldsymbol{v} ds = \boldsymbol{0} \tag{17}$$

which we recognize as the weak form of the equilibrium equations and we can, again, conclude that minimization of the potential Π results in the weak form