

The strain energy is given by

$$U = \frac{K}{2} \left( \frac{1}{2} (J^2 - 1) - \ln J \right) + \frac{G}{2} (J^{-2/3} C_{pp} - 3) \quad (1)$$

Calculate the stress response

$$S_{ij} = 2 \frac{\partial U}{\partial C_{ij}} = \frac{K}{2} (J^2 - 1) C_{ij}^{-1} + G J^{-2/3} \left( \delta_{ij} - \frac{C_{pp}}{3} C_{ij}^{-1} \right) \quad (2)$$

where it was used that

$$\frac{\partial J}{\partial C_{ij}} = \frac{J}{2} C_{ij}^{-1} \quad (3)$$

Calculate the material stiffness

$$\begin{aligned} D_{ijkl} &= 4 \frac{\partial^2 U}{\partial C_{ij} \partial C_{kl}} \\ &= a_1 C_{ij}^{-1} C_{kl}^{-1} - a_2 (\delta_{ij} C_{kl}^{-1} + C_{ij}^{-1} \delta_{kl}) + a_3 (C_{ik}^{-1} C_{jl}^{-1} + C_{il}^{-1} C_{jk}^{-1}) \end{aligned} \quad (4)$$

where

$$\begin{aligned} a_1 &= K J^2 + \frac{2G}{9} J^{-2/3} C_{pp} \\ a_2 &= \frac{2G}{3} J^{-2/3} \\ a_3 &= \frac{G}{3} J^{-2/3} C_{pp} - \frac{K}{2} (J^2 - 1) \end{aligned}$$

where it was used that

$$\frac{\partial C_{ij}^{-1}}{\partial C_{kl}} = -\frac{1}{2} (C_{ik}^{-1} C_{lj}^{-1} + C_{il}^{-1} C_{jk}^{-1}) \quad (5)$$

The matrix for plane problem becomes

$$\mathbf{D} = \begin{bmatrix} D_{1111} & D_{1122} & D_{1112} \\ D_{2211} & D_{2222} & D_{2212} \\ D_{1211} & D_{1222} & D_{1212} \end{bmatrix} \quad (6)$$