Computational Inelasticity FHLN05

Assignment 2019

A non-linear elasto-plastic wellborne drilling problem

General instructions

A written report should be submitted to the Division of Solid Mechanics no later than **November 11 at 10.00**, both a printed version and a digital version should be handed in. The digital version is sent via e-mail to *axel.henningsson@solid.lth.se*.

The assignment serves as a part exam, thus help with coding and debugging will not be provided. A maximum of 5 points can be obtained. The task can be solved individually or in groups of two. If two students work together they will obtain the same amount of points.

The report should be clear and well-structured and contain a description of the problem as well as the solution procedure; including necessary derivations and the results from the calculations in form of illustrative figures and tables. The program code should be included as an appendix. It should be sufficient with 15 pages, appendix excluded.

It can be assumed that the reader posses basic knowledge of Solid Mechanics but it has been a while since he/she dealt with this type of analysis.

After reading the report, the reader should be able to reproduce the results just by reading through the report, i.e. without using the included program. This implies that all derivations of necessary quantities such as stiffness tensor etc. should be presented in some detail.

Note, a report should be handed in even if you are not able to solve all tasks or if your program does not work!

Problem description

A wellbore drilling operation in search of water is planned to be performed. As depicted in Figure 1, the borehole is to be drilled as a perfect cylinder in a rock like structure, introducing an internal pressure p for wall support. The surrounding rock formation is in its' natural state under a compressive stress at each point.

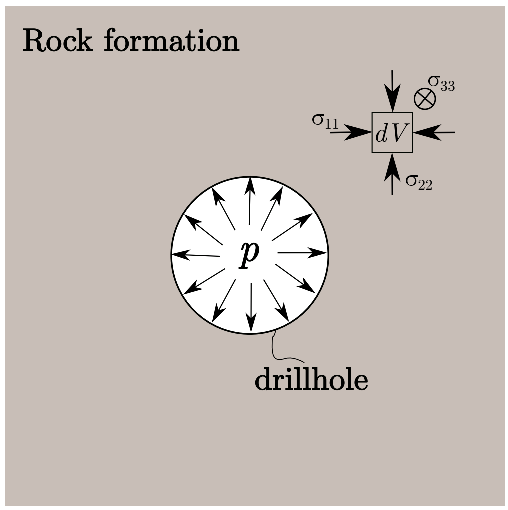

Figure 1: A drill hole is held in equilibrium by the internal pressure p. Each volume element, dV, in the body is under compressive stress.

To understand how the internal wall pressure relates to borehole contraction and potential collapse, the excavation process is to be simulated using an elasto-plastic Finite Element Model (FEM) together with a classical Newton-Raphson scheme. To simplify the procedure, a limited region around the borehole is included in the model, using **plane strain** conditions to reduce the problem to two dimensions. The resulting boundary value problem can be viewed in Figure 2A. Note that due to symmetry it is not necessary to model the entire structure, but rather one fourth of the borehole needs to be considered, as depicted in Figure 2B. Note also that the thickness, t, of the geometry is arbitrary due to the plane strain assumption. However, for practical reasons, a thickness of 1 m is to be selected.

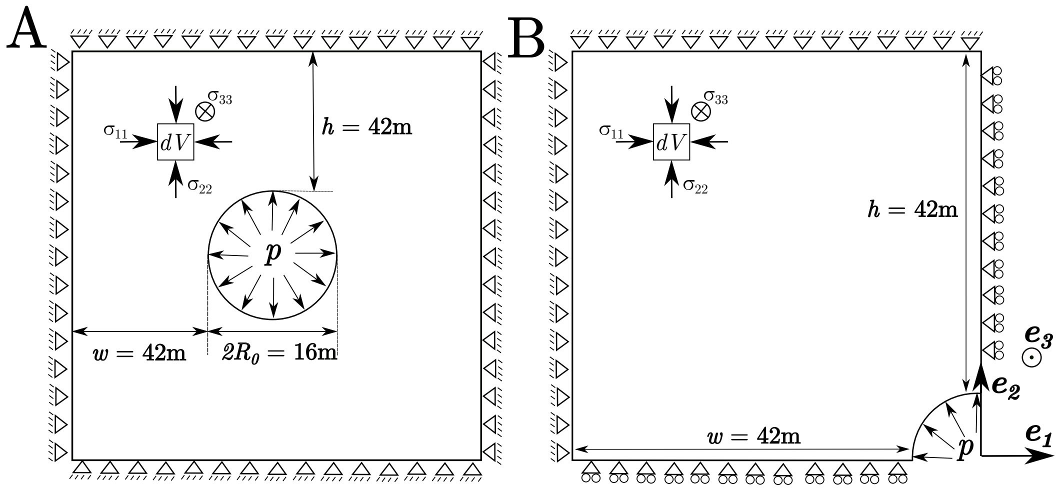

Figure 2: The edges of the rock formation is fixed in space from displacing in A. In B the symmetry is used and the boundary conditions on the right and bottom boundary is gives freedom in $\boldsymbol{e}_2$ and $\boldsymbol{e}_1$ respectively. The thickness of the rock formation, in the $\boldsymbol{e}_3$ direction, is to be taken as 1 m.

The task is to calculate the elasto-plastic response of the rock formation during unloading of the supporting wall pressure, p. To solve the problem the CALFEM-toolbox should be used. In CALFEM, certain general FE-routines are already established but you need to establish extra routines in order to solve the elastic-plastic boundary value problem. A good starting point is the code you developed in the computer lab. Three-node triangle elements are used for the finite element calculations. The mesh can be generated in Matlab using the `pdetool` GUI as we will return to in a bit.

For simplicity the rock formation is assumed to be both homogeneous and isotropic, and the elastic response, $\epsilon_{kl}^{(e)}$, of the rock formation is modelled linearly using Hooke's law $\sigma_{ij} = D_{ijkl}\epsilon_{kl}^{(e)}$ where

$$D_{ijkl} = 2G\Big[\frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \frac{\nu}{1-\nu}\delta_{ij}\delta_{kl}\Big], \qquad G = \frac{E}{2(1+\nu)} \tag{1}$$

where E and ν are the Young's modulus and Poisson's ratio, respectively and G is the shear modulus.

To model the plastic response a Drucker-Prager strain-hardening model is to be used, defining the current yield surface, f, as

$$f = \sqrt{3J_2} + \alpha I_1 + \beta \tag{2}$$

where J_2 and I_1 are the stress invariants,

$$J_2 = \frac{1}{2} s_{ij} s_{ji}, \quad I_1 = \sigma_{ii}, \tag{3}$$

and σ_{ij} and s_{ij} are the Cauchy stress tensor and deviatoric stress tensor, respectively. The parameters α and β in equation (2) are taken as

$$\alpha = \alpha(\epsilon_d^p) = \frac{\epsilon_d^p\big(\tan(\gamma_f) - \tan(\gamma_i)\big)}{3(\epsilon_d^p + c)} + \frac{1}{3}\tan(\gamma_i), \qquad \beta = 0 \tag{4}$$

where ϵ_d^p is the accumulated deviatoric effective plastic strain,

$$\epsilon_d^p = \int \dot{\epsilon}_d^p dt, \qquad \dot{\epsilon}_d^p = \sqrt{\frac{3}{2}\dot{e}_{ij}^p \dot{e}_{ji}^p}, \tag{5}$$

and e_{ij}^p is the deviatoric plastic strain. Here the constant material parameters γ_f, γ_i and c are selected to mimic the rock behaviour, and numerical values can be found in table 1.

Table 1: Material parameters.

G [MPa]	ν [-]	γ_i [^{o}]	γ_f [^{o}]	c [-]
300	0.3	30	42	0.01

To determine the evolution of plastic strain, $\dot{\epsilon}_{ij}^{(p)}$, for simplicity an associated plasticity is assumed

$$\dot{\epsilon}_{ij}^{(p)} = \dot{\lambda}\frac{\partial f}{\partial \sigma_{ij}} \tag{6}$$

where $\dot{\lambda}$ is the plastic multiplier. Especially, the evolution law can be recast as

$$\dot{\epsilon}_d^p = \dot{\lambda} \tag{7}$$

Note: in this plasticity formulation the so called *hardening parameter* and *internal variable*, denoted K and κ in the course book, is selected as α and ϵ_d^p:

$$K(\kappa) = \alpha(\kappa), \qquad \dot{\kappa} = \dot{\epsilon}_d^p, \tag{8}$$

The evolution law is however **not** given in the common format

$$\dot{\kappa} \neq \dot{\lambda}\frac{\partial f}{\partial K} = \dot{\lambda}\frac{\partial f}{\partial \alpha} \tag{9}$$

but **instead** we have that

$$\dot{\kappa} = \dot{\epsilon}_d^p = \dot{\lambda} \tag{10}$$

To be able to use the above relation in a FE-setting a useful format must be determined. This is found by integration of the evolution laws, a fully implicit scheme should be used. Thus during plastic loading, when the strain increment results in a stress state lying on the outside of the yield surface, the above flow rules must be integrated to find an appropriate step $\Delta\lambda$ used to correct the stress and return to the yield surface. The increment, $\Delta\lambda$, is further used to update the internal variable, κ, which results in a change of the yield surface. In the Appendix important help to perform and implement these derivations are given.

To achieve a quadratic convergence in the Newton-Raphson scheme, the appropriate algorithmic tangent stiffness must be computed. Note again, as specified in the box above, that the general procedure on pages 497-499 in the course book does not apply, since

$$\dot{\kappa} \neq \dot{\lambda}\frac{\partial f}{\partial K} \tag{11}$$

Setting up the boundary value problem

Before the elasto-plastic borehole boundary value problem depicted in Figure 2 can be solved, the effect of the experienced stress state of the rock formation must be established as well as the wall pressure needed for equilibrium to exist after excavation. To do this we will first solve two linear FEM problems denoted **problem 1** and **problem 2**.

In **problem 1** we will model an elastic undisturbed rock formation, introducing a compressive stress tensor to each element in the FEM mesh. The resulting boundary displacement field can then be used as boundary conditions for **problem 2**. In practice two seperate meshes can be constructed using the Matlab pdetool.

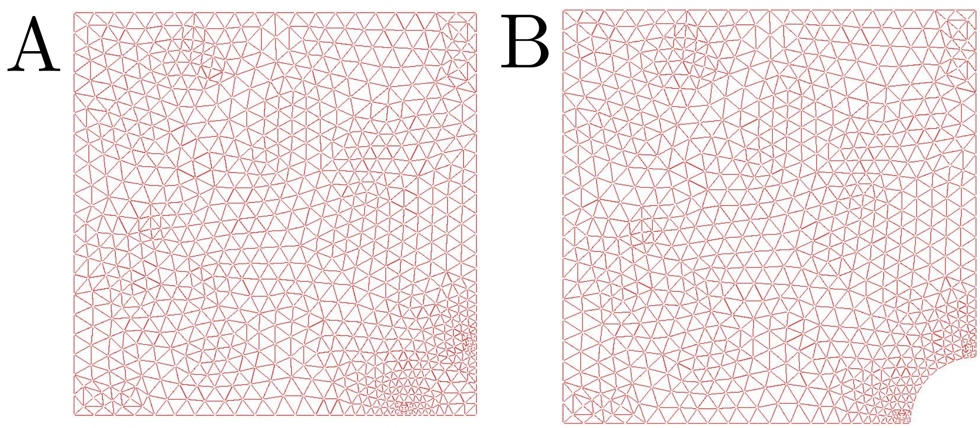

Figure 3: Two meshes are used to set up the boundary value problem. A is used in **problem 1** and the mesh B is used in **problem 2** as well as the final elasto-plastic problem.

Note that the geometry in Figure 3B must include the same elements as A, only removing the borehole elements. This can be realised by either adding or subtracting the same circular shape to a square in the Matlab pdetool as seen in Figure 4. Make sure that the mesh looks the same outside the borehole for the two geometries.

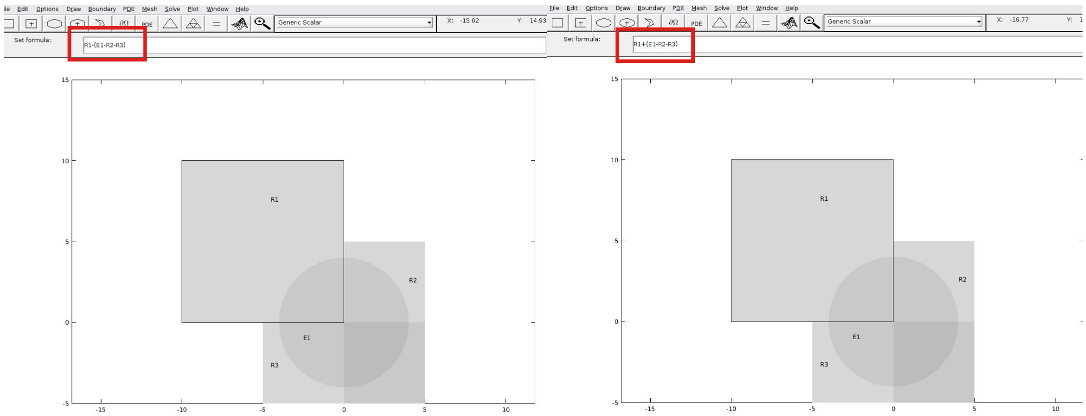

Figure 4: The Matlab pdetool is to be used in order to generate the required geometries.

We now proceed to solve the two initial linear FEM problems one at the time.

Problem 1

The geometry representing the untouched soil (Figure 3A) is prescribed, the appropriate symmetry boundary conditions together with an initial uniform stress field as shown below

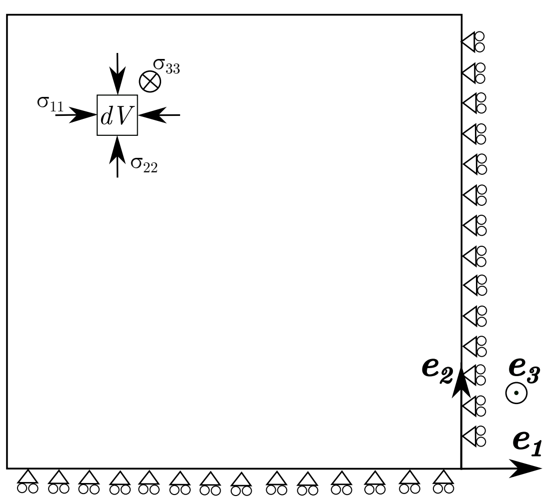

Figure 5: The untouched soil is modelled with a compressive stress to find the traction force at the drillhole boundary and the initial displacements of problem 2. Note that only one quarter of the body is modelled, hence the symmetry boundary conditions.

The stress tensor has been selected to ensure that the resulting stress state is located inside the Drucker-Prager yield surface (see pages 163-164 in the course book), i.e the elastic domain. This is the reason that the problem described in Figure 5 can be solved using linear FEM, i.e no plasticity model nor Newton-Raphson scheme is needed. The initial stress should be selected as

$$\boldsymbol{\sigma} = \begin{bmatrix} -11.25 & 0 & 0 \\ 0 & -11.25 & 0 \\ 0 & 0 & -6.75 \end{bmatrix} \text{ MPa} \tag{12}$$

Assemble the global linear stiffness matrix using Hook's law, and the internal force vector resulting from the uniform stress state. Solve for the resulting displacement field in the rock formation and save the displacements present on the **left and top boundary**. i.e the free boundaries.

To compute the pressure needed for balance at the drill hole wall, one may assemble the internal forces for **only** the elements which exist inside the radius of the borehole. The internal force present at the drill hole boundary nodes can then be extracted and saved. Remember that a change of sign is needed for the pressure to be directed towards the borehole wall.

Problem 2

Using the displacements and forces calculated as described above, we may now set up the initial state of the drilled geometry (B). By applying the pressure to the wall nodes, as found from **problem 1**, and enforcing the prescribed displacements to the free boundaries (left and top) we can solve yet another linear FEM problem. Note that the internal stress state now is inferred by the boundary conditions and the applied force. This means that the element stress should be set to zero in **problem 2**. The problem is depicted in Figure 6.

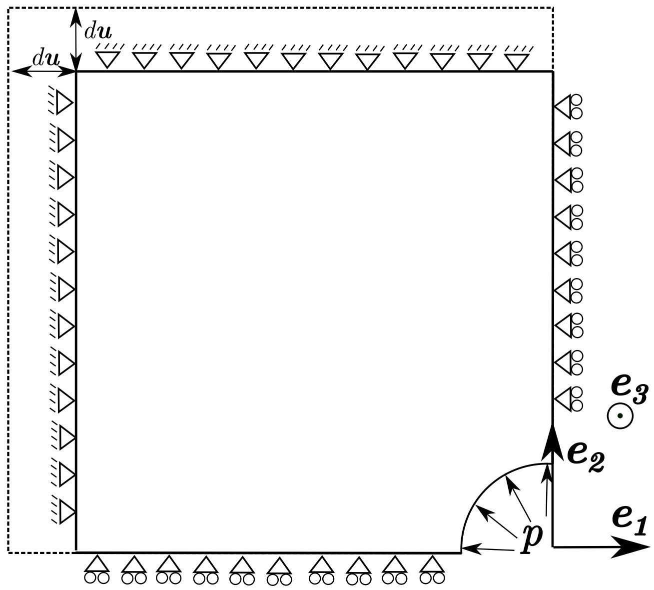

Figure 6: By setting the stress in each volume element to zero and enforcing a boundary displacement together with the pressure p the initial state of the elasto-plastic problem is retrieved.

Remember that the two geometries in Figure 3A and B has different number of elements and nodes, thus the degrees of freedom and node numbers does not necessarily refer to the same thing in A as in B. A simple way to transfer a quantity from one geometry to the other is to loop over the nodes and check if their coordinates match.

Hint: Given the **p**,**e** and **t** matrices from Matlabs pdetool, the classical FEM quantities can be computed as

```
nen=3;        % number of nodes per element
enod=t(1:nen,:)';    % element nodes matrix
nelm=size(enod,1);   % number of elements
coord=p';        % nodal coordinates
dofnode=2;       % number of degrees of freedom per node
nnod=size(coord,1);     % number of nodes
dof=[(1:nnod)',(nnod+1:2*nnod)'];        % give each dof a number
ndof=max(max(dof));     % number of degrees of freedom
edof=zeros(nelm,nen*dofnode+1);     % allocate space for edof

% Generate edof from enod and dof
for ie=1:nelm
   edof(ie,:)=[ie dof(enod(ie,1),:), dof(enod(ie,2),:), ...
       dof(enod(ie,3),:)];
end
```

If the above procedure has been performed correctly the same stress tensor as prescribed to the elements in **problem 1** should be found in the elements after solving **problem 2**. Compute the strains and stresses resulting from the displacements in **problem 2** and check that the corresponding stress is the same as the one you applied in **problem 1**.

Elasto-plastic drillhole problem.

We are now ready to solve our elasto-plastic problem. Use the displacement field from **problem 2** as the initial state for the configuration. The left and top boundaries will then have an initial displacement. Thus in the continuing calculations the displacement boundary condition at the left and top boundaries should be zero. When the Newton-Raphson equilibrium loop starts, there should exist already an internal force vector, previous strain, and previous displacement field, i.e an initial state. The final elasto-plastic boundary value problem to be solved is depicted in Figure 7.

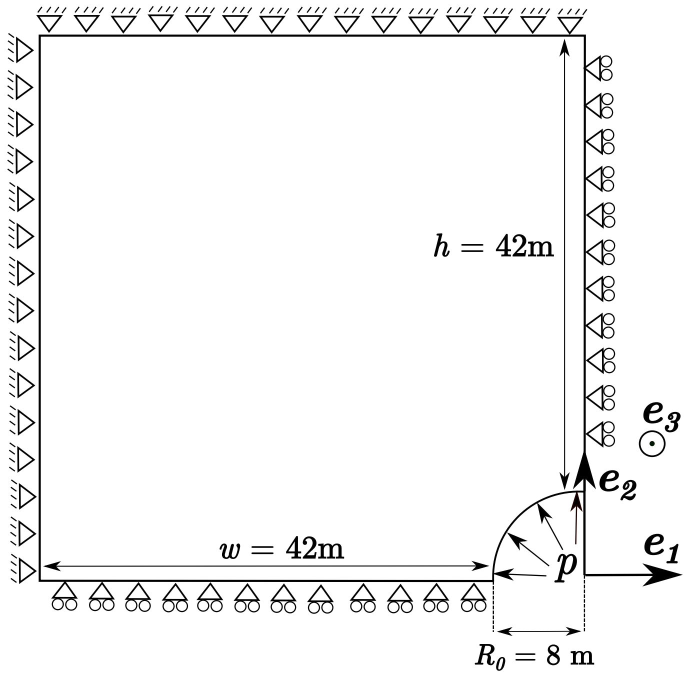

Figure 7: The final elasto-plastic problem is solved using a Newton-Raphson iteration scheme. The quantities retrieved from **problem 2** are refereed to as the previous equilibrium state.

The assignment includes the following tasks

- Derive the FE formulation of the equations of motion.
- Derive the equilibrium iteration procedure by defining and linearizing a residual, i.e. the Newton-Raphson procedure.
- Derive the numerical algorithmic tangent stiffness $\mathbf{D_{ats}}$ and the stress correction formula needed for the presented Drucker-Prager plasticity model.
- Implement the subroutine `update_variables.m` that checks for elasto-plastic response and updates accordingly (a manual for the routines is appended).
- Implement of the subroutine `alg_tan_stiff.m` that calculates the algorithmic tangent stiffness (a manual for this routine is appended).
- Unload the internal wall pressure to 30% of the initial pressure, p. A reasonable step-size is 1% of the initial pressure, p per step. Produce the following plots:

1. Plot the load path in $I_1 - \sqrt{3J_2}$-space for the element experiencing the greatest (most positive) hydro-static stress state, I_1 (closest to Drucker-Prager cone tip). Include the two lines that traces the initial and final yield surface for the selected element.
2. Plot the drill hole normalised radius R_0/r as a function of wall pressure. Here R_0 is the initial radius and r the current radius.
3. Plot the effective von Mises stress field in the body after unloading, i.e plot $\sqrt{3J_2}$.
4. Plot the volumetric stress field in the body after unloading, i.e plot I_1.
5. Plot the the deformed structure on top of the original structure, illustrating the deformation. Zoom in to the borehole and use magnification if needed.
6. Plot the mesh and indicate which elements responded plastic at any point during unloading. Preferably two different colours are used to illustrate linear and elasto-plastic elements.

The report should be well structured and contain sufficient details of the derivations with given assumptions and approximations for the reader to understand. Furthermore, some useful hints are given in the appendix.

Some interesting questions to consider are:

- Are the results reasonable?
- If the mesh is refined (or made coarser) does it change the results?
- What are the limitations for application of the approach, i.e, what are the main assumptions?
- What would happen if unloading to 0% or close to 0% of the initial pressure was performed. Could this be modelled with the current Drucker-Prager formulation?
- What is the physical interpretation of the angles γ_i and γ_f? What does it mean to change these values?

Good luck!

Appendix

Hints to derivation of stress correction

Note that you will have to complete the missing steps in the proof below for the report.

The return method is realised by considering that the elastic strain, $\epsilon_{ij}^{(e)}$, present during a single iteration step is computed from the trial strain, $\epsilon_{ij}^{(tr)}$, by subtracting the plastic increment in strain $\Delta\epsilon_{ij}^{(p)}$ as

$$\epsilon_{ij}^{(e)} = \epsilon_{ij}^{(tr)} - \Delta\epsilon_{ij}^{(p)} \tag{13}$$

Using the associated plasticity we find

$$\epsilon_{ij}^{(e)} = \epsilon_{ij}^{(tr)} - \Delta\lambda \frac{\partial f^{(2)}}{\partial \sigma_{ij}} = \epsilon_{ij}^{(tr)} - \Delta\lambda \left(\alpha\delta_{ij} + \frac{3}{2} \frac{s_{ij}^{(2)}}{\sqrt{3J_2^{(2)}}} \right) \tag{14}$$

here superscript (2) indicates that the quantity is evaluated at the new state being calculated for. Splitting equation (14) into elastic deviatoric strain, $e_{ij}^{(e)}$, and elastic volumetric strain, $\epsilon_{kk}^{(e)}$, we have

$$\begin{aligned} \epsilon_{kk}^{(e)} &= \epsilon_{kk}^{(tr)} - 3\Delta\lambda\alpha \\ e_{ij}^{(e)} &= e_{ij}^{(tr)} - \Delta\lambda \frac{3}{2} \frac{s_{ij}^{(2)}}{\sqrt{3J_2^{(2)}}} \end{aligned} \tag{15}$$

Hook's law states that for elastic strain

$$\sigma_{kk} = 3K\epsilon_{kk}^{(e)}, \qquad s_{ij} = 2Ge_{ij}^{(e)} \tag{16}$$

where G is the shear modulus as previously and K here is the bulk modulus

$$K = \frac{E}{3(1-2\nu)} \tag{17}$$

(see page 91 in the course book). By multiplying equation (15) with $2G$ and $3K$ it is possible to obtain

$$\begin{aligned} I_1^{(2)} &= I_1^{(tr)} - 9K\Delta\lambda\alpha \\ s_{ij}^{(2)} &= s_{ij}^{(tr)} - 3G\Delta\lambda \frac{s_{ij}^{(2)}}{\sqrt{3J_2^{(2)}}} \end{aligned} \tag{18}$$

Considering (18) and making use of that

$$\sqrt{3J_2^{(2)}} = \sqrt{\frac{3}{2}s_{ij}^{(2)}s_{ji}^{(2)}} \tag{19}$$

it is possible to arrive at

$$\frac{s_{ij}^{(tr)}}{\sqrt{3J_2^{(tr)}}} = \frac{s_{ij}^{(2)}}{\sqrt{3J_2^{(2)}}} \tag{20}$$

Inserting this in (18) we find that the updated deviatoric stress is

$$s_{ij}^{(2)} = s_{ij}^{(tr)} - 3G\Delta\lambda\frac{s_{ij}^{(tr)}}{\sqrt{3J_2^{(tr)}}} \tag{21}$$

by addition of the updated hydrostatic stress $\frac{1}{3}I_1^{(2)}\delta_{ij}$ we finaly arrive at

$$\sigma_{ij}^{(2)} = s_{ij}^{(tr)} - 3G\Delta\lambda\frac{s_{ij}^{(tr)}}{\sqrt{3J_2^{(tr)}}} + \frac{1}{3}(I_1^{(tr)} - 9K\Delta\lambda\alpha)\delta_{ij} \tag{22}$$

Hints to calculation of increment $\Delta\lambda$

Note that you will have to complete the missing steps in the proof below for the report.

To compute $\Delta\lambda$ consider the equation

$$\alpha = \alpha(\epsilon_d^p) = \frac{\epsilon_d^p\big(\tan(\gamma_f) - \tan(\gamma_i)\big)}{3(\epsilon_d^p + c)} + \frac{1}{3}\tan(\gamma_i) \tag{23}$$

We may rewrite such that ϵ_d^p is a function of α:

$$3\alpha\epsilon_d^p + 3\alpha c = \epsilon_d^p\big(\tan(\gamma_f) - \tan(\gamma_i)\big) + \tan(\gamma_i)(\epsilon_d^p + c) \Rightarrow \tag{24}$$

$$3\alpha\epsilon_d^p + 3\alpha c - \epsilon_d^p\big(\tan(\gamma_f) - \tan(\gamma_i)\big) - \tan(\gamma_i)\epsilon_d^p = \tan(\gamma_i)c \Rightarrow \tag{25}$$

$$\epsilon_d^p(3\alpha - \big(\tan(\gamma_f) - \tan(\gamma_i)\big) - \tan(\gamma_i)) = \tan(\gamma_i)c - 3\alpha c \Rightarrow \tag{26}$$

$$\epsilon_d^p = \frac{\tan(\gamma_i)c - 3\alpha c}{3\alpha - \tan(\gamma_f)} \tag{27}$$

The incremental relation becomes

$$\dot{\epsilon}_d^p = \frac{-3\dot{\alpha}c(3\alpha - \tan(\gamma_f)) - 3\dot{\alpha}(\tan(\gamma_i)c - 3\alpha c)}{(3\alpha - \tan(\gamma_f))^2} \Rightarrow \tag{28}$$

$$\dot{\epsilon}_d^p = \frac{3c(\tan(\gamma_f) - \tan(\gamma_i))}{(\tan(\gamma_f) - 3\alpha)^2}\dot{\alpha} \tag{29}$$

Integrating this relation between α_n and α and using equation (7) can provide

$$\alpha = \frac{1}{3}\tan(\gamma_f) + \frac{1}{3}\Big[\frac{1}{3\alpha_n - \tan(\gamma_f)} - \frac{\Delta\lambda}{c(\tan(\gamma_f) - \tan(\gamma_i))}\Big]^{-1} \tag{30}$$

where α_n is the value of α at the previous equilibrium state. Now we use the two equations in (18) together with (30) to find an equation with $\Delta\lambda$ as the only unknown. For plastic loading this equation is to be solved under the constraint of

$$\Delta\lambda \geq 0; \quad \Delta\lambda f(\sigma_{ij}) = 0 \tag{31}$$

i.e we require that the increment satisfies $f(\sigma_{ij}) = 0$. By expressing the yield function in terms of $\Delta\lambda$ and solving for $\Delta\lambda$ with a numerical solver such as Matlabs `fzero` function the increment is found.

Hints to derivation of $\boldsymbol{D}_{ats}$

The algorithmic tangent stiffness, $\boldsymbol{D}_{ats}$, is defined as the derivative at the new position denoted (2):

$$d\boldsymbol{\sigma}^{(2)} = \boldsymbol{D}_{ats} d\boldsymbol{\epsilon}^{(2)} \tag{32}$$

With a fully implicit scheme, we have the following equations

$$\begin{aligned}
&\boldsymbol{\sigma}^{(2)} = \boldsymbol{\sigma}^{t} - \boldsymbol{D}\Delta\boldsymbol{\epsilon}^{p} \\
&\boldsymbol{\sigma}^{t} = \boldsymbol{\sigma}^{(1)} + \boldsymbol{D}\Delta\boldsymbol{\epsilon} \\
&\Delta\boldsymbol{\epsilon}^{p} = \Delta\lambda\Big(\frac{\partial\widehat{f}(\boldsymbol{\sigma}, K)}{\partial\boldsymbol{\sigma}}\Big)^{(2)} \\
&\Delta\kappa = \Delta\lambda \\
&K^{(2)} = K(\kappa^{(2)}) \\
&f(\boldsymbol{\sigma}^{(2)}, K^{(2)}) = 0
\end{aligned} \tag{33}$$

From these equations the $\boldsymbol{D}_{ats}$ can be derived. Note that the matrix definition of strain in 3D is given by

$$\Delta\boldsymbol{\epsilon}^{p} = \begin{bmatrix}\Delta\epsilon_{11}^{p} & \Delta\epsilon_{22}^{p} & \Delta\epsilon_{33}^{p} & 2\Delta\epsilon_{12}^{p} & 2\Delta\epsilon_{13}^{p} & 2\Delta\epsilon_{23}^{p}\end{bmatrix}^{T}$$

Furthermore, note that $\partial\widehat{f}/\partial\boldsymbol{\sigma}$ is a function of both the stress, $\boldsymbol{\sigma}$, and the hardening parameter K.

alg_tan_stiff

Purpose: Compute the algorithmic tangent stiffness matrix for a triangular 3 node element under plane strain conditions for the above presented Drucker-Prager plasticity model.

Syntax: `Dats = alg_tan_stiff(sigma,dlambda,ep_eff,Dstar,mp)`

Description: `alg_tan_stiff` provides the algorithmic tangent stiffness matrix `Dats` for a triangular 3 node element. The stress is provided by `sigma`

$$\texttt{sigma} = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \end{bmatrix}$$

`Dstar` is the linear elastic material tangent for plane strain, `dlambda` is the increment $\Delta\lambda$, `ep_eff` is the effective deviatoric plastic strain ϵ_d^p and `mp` a vector containing the material parameters needed.

update_variables

Purpose: Check for elasto-plastic response and update variables accordingly for a triangular 3 node element under plane stress conditions for the presented Drucker-Prager plasticity model.

Syntax:

```
[sigma,dlambda,ep_eff] =

update_variables(sigma_old,ep_eff_old,delta_eps,Dstar,mp)
```

Description: `update_variables` provides updates of the stress `sigma`, the increment in plastic multiplier `dlambda` and the effective plastic deviatoric strain `ep_eff`. The variables are calculated from stress and effective plastic strain at the last accepted equilibrium state `sigma_old` and `ep_eff_old`, respectively. The increment in strains between the last equilibrium state and the current state; `delta_eps`.

The increment $\Delta\lambda$ needed to update the stresses and strains are also computed and could be used as an indicator of plasticity later on in the code and will therefore also be used as output from this function.

Moreover `Dstar` denotes the linear elastic material tangent for plane strain and `mp` is a vector containing the material parameters needed.