Computational Inelasticity FHLN05

Assignment 2018

A non-linear elasto-plastic problem

General instructions

A written report should be submitted to the Division of Solid Mechanics no later than **November 5 at 10.00**, both a printed version and a digital version should be handed in.

The digital version is sent via e-mail to *marcus.alexandersson@solid.lth.se.*

The assignment serves as a part exam, thus help with coding and debugging will not be provided. A maximum of 5 points can be obtained. The task can be solved individually or in groups of two. If two students work together they will obtain the same amount of points.

The report should be clear and well-structured and contain a description of the problem as well as the solution procedure; including necessary derivations and the results from the calculations in form of illustrative figures and tables. The program code should be included as an appendix. It should be sufficient with 15 pages, appendix excluded.

It can be assumed that the reader posses basic knowledge of Solid Mechanics but it has been a while since he/she dealt with this type of analysis.

After reading the report, the reader should be able to reproduce the results just by reading through the report, i.e. without using the included program. This implies that all derivations of necessary quantities such as stiffness tensor etc. should be presented in some detail.

Note, a report should be handed in even if you are not able to solve all tasks or if your program does not work!

Problem description

An aluminium detail consists of two different parts made from sheet metal of 1 mm thickness. The geometry of the detail is seen in Figure 1.

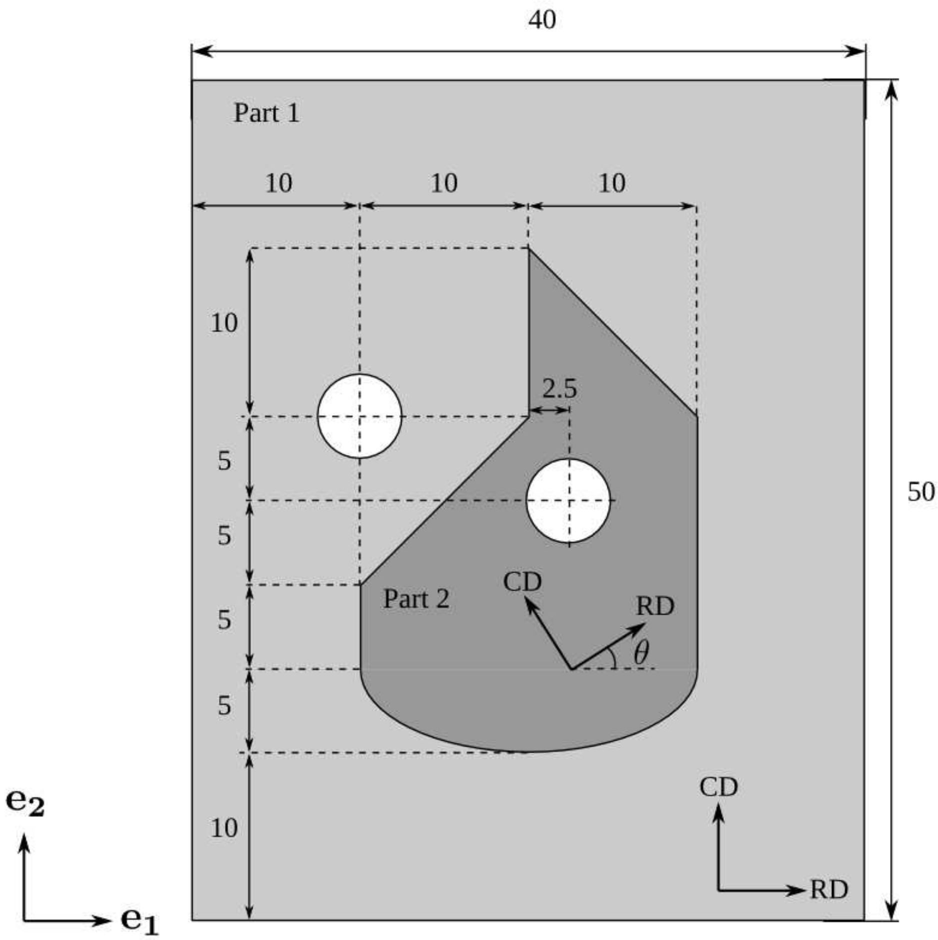

Figure 1: The geometry of the detail composed of two part with different material orientations. The holes both have diameter 5 mm. Note that all dimensions are given in mm.

As a consequence of the cold rolling processes used to obtain the sheet material the parts are orthotropic with the directions as rolling direction (RD), cross-rolling direction (CD) and out-of-plane direction (ZD) as the symmetry axis for the respective sheets. In a later stage of the manufacturing the detail is strained in

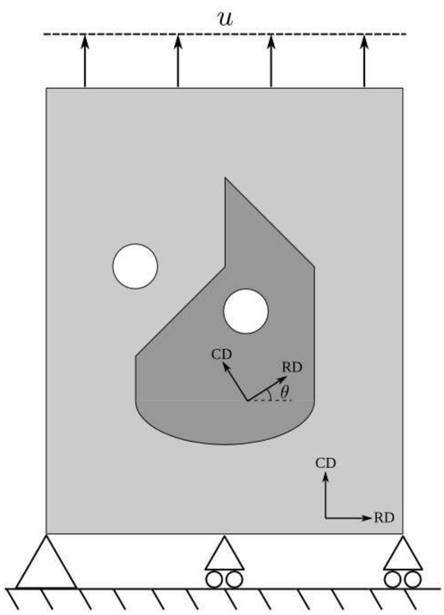

Figure 2: Schematic sketch of the boundary conditions.

uniaxial tension along CD of the outer part (part 1), i.e along $\boldsymbol{e}_2$. The boundary conditions can be seen in figure 2.

For simplicity is assumed that the elastic properties of the material is isotropic and only the plasticity is anisotropic. The small thickness of the detail allow for the simplification of assuming a **plane stress state**. Thus Hooke's law for plane stress can be used to model the elasticity. The plastic response of the material is modelled using the Hill yield surface with isotropic hardening in order to account for the orthoropic nature of the material:

$$f = \sqrt{\sigma_{y0}^2 \boldsymbol{s}^T \boldsymbol{P} \boldsymbol{s}} - \sigma_y = 0, \quad \sigma_{eff} = \sqrt{\sigma_{y0}^2 \boldsymbol{s}^T \boldsymbol{P} \boldsymbol{s}} \tag{1}$$

where associated plasticity can be assumed. The P-matrix for the Hill yield surface under plane stress condition is given as

$$\boldsymbol{P} = \begin{bmatrix} F+G & -F & -G & 0 \\ -F & F+H & -H & 0 \\ -G & -H & G+H & 0 \\ 0 & 0 & 0 & 2L \end{bmatrix}, \quad \boldsymbol{s} = \begin{bmatrix} s_{11} \\ s_{22} \\ s_{33} \\ s_{12} \end{bmatrix}$$

where F, G, H and L are material constants related to the material data in table 1 (relations given in appendix). The yield stress for the material is given as

$$\sigma_y = \sigma_{y0} + k\sigma_{y0}(\varepsilon_{eff}^p)^n$$

E [GPa]	ν [-]	$\sigma_{y0}^{(RD)}$ [MPa]	$\sigma_{y0}^{(CD)}$ [MPa]	$\sigma_{y0}^{(ZD)}$ [MPa]	k [-]	n [-]
70	0.32	450	250	250	5	0.45

Table 1: Material data.

The task is to calculate the elasto-plastic response of the detail. The elasto-plastic response is the solution to the equation of motion (static conditions may be assumed and body forces may be neglected). To solve the problem the CALFEM-toolbox should be used. In CALFEM, certain general FE-routines are already established but you need to establish extra routines in order to solve the elastic-plastic boundary value problem.

For the global equilibrium loop a Newton-Raphson scheme should be implemented and for the integration of the elasto-plastic constitutive laws a fully implicit radial return method should be used (cf. chapter 18 in the course book, note that plane stress conditions prevail!). Three-node triangle elements are used for the finite element calculations. The mesh can be generated in Matlab using the `pdetool` GUI.

The assignment includes the following

- Derive the FE formulation of the equation of motion.
- Derive the equilibrium iteration procedure by defining and linearizing a residual, i.e. Newton-Raphson procedure.
- Derive the numerical algorithmic tangent stiffness $\mathbf{D}_{\mathbf{ats}}$ and the radial return method for isotropic hardening of the Hill yield surface.
- Use force controlled loading of a simple structure (Figure 3) to obtain the uniaxial response (loading and unloading) of the material loaded in RD and CD.

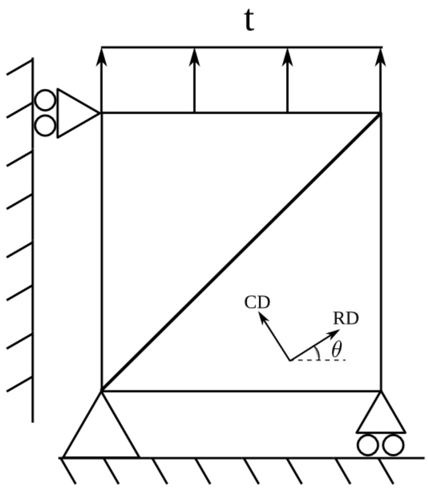

Figure 3: Simple mesh for unixial tension with traction $\boldsymbol{t}$.

- Investigate the elasto-plastic response of the detail with part 2 material at different rotation angle θ by implementing a FE program using the Newton-Raphson algorithm with a fully implicit radial return method using **displacement control**. This includes:
  - Implementation of the subroutines `update_variables.m` that checks for elasto-plastic response and updates accordingly (a manual for the routines is appended). The routines can be checked with data from `check_update_assignment2018.mat`.
  - Implementation of the subroutines `alg_tan_stiff.m` that calculates the algorithmic tangent stiffness (a manual for this routine is appended) of the corresponding material. The routines can be checked with data from `check_Dats_assignment2018.mat`.
- Use displacement controlled loading and stretch the top boundary 0.2 mm (correspond to $\sigma^{(CD)}$ of part 1) then return to the original position. Investigate the peak stresses and the residual stress as well as the evolution of the plastic region(s) during this process.
- The following results should be presented in an illustrative way:
  - An σ-ε curve for loading and unloading (loaded into the plastic regime) for the particular model under uniaxial tension, (analytic expressions may be used for comparison) in RD and CD respectively.
  - The development of plastic response regions of the detail during procedure. Compare the results for the orientation of part 2; $\theta = 90°$ and one other orientation of your choice.

- The Hill effective stress distribution at maximum displacement and the residual stresses in the detail (the top displaced back to original position). Compare the data for the orientation of part 2; $\theta = 0°$ and $\theta = 90°$ as well as one other case of your choice, e.g. any other orientation angle or a special case of isotropic material in part 2 (can be obtained by annealing the material before insertion into part 1).

The report should be well structured and contain sufficient details of the derivations with given assumptions and approximations for the reader to understand. Furthermore, some useful hints are given in appendix.

Some interesting questions to consider are:

- Are the results reasonable?
- If the mesh is refined does it change the results?
- What are the limitations for application of the approach, i.e, what are the main assumptions?
- Is there a significant difference between the peak stresses, residual stresses and/or the plasticity for various orientation of the material of the insert?
- High residual stresses can typically be problematic in some applications. Based on your analysis should the manufacturer be more careful with the orientation of part 2 even though the material cost could increase due to waste or can it perhaps be used to their advantage?
- Where are the stress concentrations found? How would you suggest to improve the design?

Good luck!

Appendix

Material constants

The material constant σ_{y0} is defined as

$$\sigma_{y0} = \Big(\frac{3}{2(F+G+H)}\Big)^{1/2}$$

The material constants F, G, H, L can be calculated from

$$F = \frac{1}{2}\Big[\frac{1}{(\sigma_{y0}^{11})^2} + \frac{1}{(\sigma_{y0}^{22})^2} - \frac{1}{(\sigma_{y0}^{33})^2}\Big]$$

$$G = \frac{1}{2}\Big[\frac{1}{(\sigma_{y0}^{11})^2} + \frac{1}{(\sigma_{y0}^{33})^2} - \frac{1}{(\sigma_{y0}^{22})^2}\Big]$$

$$H = \frac{1}{2}\Big[\frac{1}{(\sigma_{y0}^{22})^2} + \frac{1}{(\sigma_{y0}^{33})^2} - \frac{1}{(\sigma_{y0}^{11})^2}\Big]$$

$$L = \frac{3}{2}\frac{1}{(\sigma_{y0})^2}$$

Hints

1) Note that the Hill yield criterion is only valid if the coordinate system coincide with the orthotropic material directions. Thus in order to account for various orientation an *a priori* transformation of coordinate system is required. In Voigt matrix notation the matrix $\boldsymbol{L}$ represent the transformation to another coordinate system rotated the angle θ counter-clockwise around the z-axis. The following relation holds

$$\boldsymbol{\sigma}' = \boldsymbol{L}\boldsymbol{\sigma}, \quad \boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix}$$

Furthermore it is possible to introduce a matrix that maps stress to its deviatoric stress as

$$\boldsymbol{s} = \boldsymbol{T}\boldsymbol{\sigma}$$

Identify chec$\boldsymbol{L}$ and $\boldsymbol{T}$. Evidently the Hill criterion can be written in as a function of the rotation as

$$f = \sqrt{\boldsymbol{\sigma}^T \hat{\boldsymbol{P}}(\theta) \boldsymbol{\sigma}} - \sigma_y = 0$$

Identify $\hat{\boldsymbol{P}}(\theta)$. What is the form in the special case of isotropy?

2) From $f = f(\boldsymbol{\sigma}^{(2)}, K^{(2)}) = 0$ it is possible to derive a constraint that can be used to find the increment $\Delta\lambda$;

$$(\boldsymbol{\sigma}^t)^T \boldsymbol{M}^T \hat{\boldsymbol{P}} \boldsymbol{M} \boldsymbol{\sigma}^t - \sigma_y^2 = 0 \tag{2}$$

Note, the report should contain a derivation of this expression in order to get maximum number of points on the assignment. Note that $\boldsymbol{M}$ depends on $\Delta\lambda$! In order to solve the constraint for $\Delta\lambda$ the command `fzero` in Matlab could be used.

3) You could use a modified Newton-Raphson scheme to solve the problem, i.e. use the elastic tangent stiffness instead of $\mathbf{D}_{ats}$. The convergence will then be impaired but it could be useful when developing your program. Note that for a maximum number of points on the assignment you will need to use the full Newton-Raphson.

General tips

It is advantageous to start by solving the problem using the simple domain with isotropy (set $\sigma_{y0}^{RD} = \sigma_{y0}^{CD} = \sigma_{y0}^{ZD}$) in order to make sure the simplest cases work before adding more complexity.

If you are familiar with writing in *Latex* it is recommended to use it when writing the report since it is fast and easy to write equations. A template is provided on the course home page for those interested.

Using the correct $\boldsymbol{D}_{ats}$ should yield roughly 4-6 iterations to obtain the correct solution provided the incremental displacement is not very large (quadratic convergence hold in the neighbourhood of the solution).

alg_tan_stiff

Purpose: Compute the algorithmic tangent stiffness matrix for a triangular 3 node element under plane stress conditions for a isotropically hardening ortotropic Hill material.

Syntax: `Dats = alg_tan_stiff(sigma,dlambda,ep_eff,Dstar,mp,rotation)`

Description: `alg_tan_stiff` provides the algorithmic tangent stiffness matrix `Dats` for a triangular 3 node element. The in-plane stress is provided by `sigma`

$$\texttt{sigma} = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix}$$

`Dstar` is the linear elastic material tangent for plane stress, `dlambda` is the increment $\Delta\lambda$, `ep_eff` is the Hill effective plastic strain ε^p_{eff} and `mp` a vector containing the material parameters needed. The parameter `rotation` provides the counter-clockwise rotation angle for the RD relative to the global coordinate system. The algorithmic tangent stiffness is given as

$$\boldsymbol{D}_{ats} = \boldsymbol{D}^a - \frac{1}{A^a}\boldsymbol{D}^a \frac{\partial f}{\partial \boldsymbol{\sigma}} \Big(\frac{\partial f}{\partial \boldsymbol{\sigma}}\Big)^T \boldsymbol{D}^a$$

$$\boldsymbol{D}^a = \Big((\boldsymbol{D}^*)^{-1} + \Delta\lambda \frac{\partial^2 f}{\partial \boldsymbol{\sigma} \partial \boldsymbol{\sigma}}\Big)^{-1}$$

$$A^a = \Big(\frac{\partial f}{\partial \boldsymbol{\sigma}}\Big)^T \boldsymbol{D}^a \frac{\partial f}{\partial \boldsymbol{\sigma}} - \frac{\partial f}{\partial K} d_a$$

$$d_a = -\frac{\partial K}{\partial \kappa}\frac{\partial f}{\partial K}$$

Note that you need to prove this in the report!

update_variables

Purpose: Check for elasto-plastic response and update variables accordingly for a triangular 3 node element under plane stress conditions for a isotropically hardening ortotropic Hill material.

Syntax:

```
[sigma,dlambda,ep_eff] =

update_variables(sigma_old,ep_eff_old,delta_eps,Dstar,mp,rotation)
```

Description: `update_variables` provides updates of the in-plane stress `sigma`, the increment in plastic multiplier `dlambda` and the Hill effective plastic strain `ep_eff`. The variables are calculated from stress and effective plastic strain at the last accepted equilibrium state `sigma_old` and `ep_eff_old`, respectively. The increment in strains between the last equilibrium state and the current state; `delta_eps`.

The increment $\Delta\lambda$ needed to update the stresses and strains are also computed and could be used as an indicator of plasticity later on in the code and will therefore also be used as output from this function.

Moreover `Dstar` denotes the linear elastic material tangent for plane stress and `mp` is a vector containing the material parameters needed. The parameter `rotation` provides the counter-clockwise rotation angle for the RD relative to the global coordinate system.