HÅLLFASTHETSLÄRA, LTH

Examination in computational materials modeling

TID: 2018-11-02, kl 13.00-19.00

Maximalt 60 poäng kan erhållas på denna tenta. För godkänt krävs 30 poäng.

Tillåtet hjälpmedel: räknare

PROBLEM 1 (10 p.)

For a linear isotropic material, the stress and strain relations can be written as

$$s_{ij} = 2Ge_{ij} \qquad \sigma_{kk} = 3K\epsilon_{kk}$$

where the deviatoric stresses s_{ij}, deviatoric strains e_{ij} are defined by

$$s_{ij} = \sigma_{ij} - \frac{1}{3}\delta_{ij}\sigma_{kk}, \quad e_{ij} = \epsilon_{ij} - \frac{1}{3}\delta_{ij}\epsilon_{kk},$$

In matrix format this relation can be written as

$$\boldsymbol{\sigma} = \boldsymbol{D}\boldsymbol{\epsilon}$$

where

$$\boldsymbol{\sigma}^T = [\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{13}, \sigma_{23}]$$

and

$$\boldsymbol{\epsilon}^T = [\epsilon_{11}, \epsilon_{22}, \epsilon_{33}, 2\epsilon_{12}, 2\epsilon_{13}, 2\epsilon_{23}]$$

Using the material parameters G and K identify the *first row* in $\boldsymbol{D}$.

PROBLEM 2 (10 p.)

The coordinate system x_i is considered. If the coordinate system is rotated and translated to obtain a new x_i'-coordinate system, the relation between x_i' and x_i is given by

$$x_i' = A_{ij}(x_j - c_j)$$

where A_{ij} is the transformation matrix. This matrix has the following properties

$$A_{ki}A_{kj} = \delta_{ij}\,; \quad A_{ik}A_{jk} = \delta_{ij}$$

Let B_{ij} denote the components of a second-order tensor in the x_i-coordinate system and n_i the components of a first-order tensor. Since B_{ij} is a second-order tensor, its components B_{ij}' in the x_i'-coordinate system is given by

$$B_{ij}' = A_{ik}B_{kl}A_{jl}$$

in the same manner for the n_i, its components in the x_i'-coordinate system is given by

$$n_i' = A_{ik}n_k$$

Consider the joint invariants

$$J_1^* = n_i B_{ij} n_j, \qquad J_2^* = n_i B_{ik} B_{kj} n_j$$

Show that J_1^* and J_2^* are invariants. (Remark: these invariants can be used to describe anisotropic material behaviour).

PROBLEM 3 (10 p.)

For a general isotropic hyper elastic material the strain energy can be written in terms of the strain invariants, i.e.

$$W = W(\tilde{I}_1, \tilde{I}_2, \tilde{I}_3)$$

where

$$\tilde{I}_1 = \epsilon_{kk}, \quad \tilde{I}_2 = \frac{1}{2}\epsilon_{ij}\epsilon_{ji}, \quad \tilde{I}_3 = \frac{1}{3}\epsilon_{ik}\epsilon_{kj}\epsilon_{ji}$$

a) Determine the stress-strain relation based on

$$\sigma_{ij} = \frac{\partial W}{\partial \epsilon_{ij}}$$

and evaluate

$$\frac{\partial \tilde{I}_\alpha}{\partial \epsilon_{ij}} \qquad \alpha = 1, 2, 3$$

b) Determine that fourth-order tensor D^s_{ijkl} given by

$$\sigma_{ij} = D^s_{ijkl}\epsilon_{kl}$$

based on a) and that

$$\frac{\partial W}{\partial \tilde{I}_1} = \lambda \epsilon_{kk}$$

where λ is a constant material parameter.

c) Using the incremental relation

$$d\sigma_{ij} = D^t_{ijkl} d\epsilon_{kl} \qquad \text{where} \qquad D^t_{ijkl} = \frac{\partial^2 W}{\partial \epsilon_{ij} \partial \epsilon_{kl}}$$

To identify D^t_{ijkl} assume that

$$\frac{\partial W}{\partial \tilde{I}_2} \quad \text{and} \quad \frac{\partial W}{\partial \tilde{I}_3} \quad \text{are constants}$$

d) In a uniaxial loading situation, i.e. in a $\sigma - \epsilon$-graph, illustrate D^s and D^t (the corresponding uniaxial quantities). No calculations are necessary, a graphical illustration is sufficient.

PROBLEM 4 (10 p.)

An orthotropic yield criterion for pressure independent material is given by

$$\hat{f} = \sqrt{\sigma_{yo}^2 \boldsymbol{\sigma}^T \boldsymbol{P} \boldsymbol{\sigma}} - \sigma_y(\kappa) = 0$$

where symmetry of the stress tensor was used, $\sigma_y = \sigma_{yo} + K(\kappa)$ and $K(\kappa)$ is the hardening function depending on the internal variable κ. Moreover

$$\boldsymbol{P} = \begin{bmatrix} F+G & -F & -G & 0 & 0 & 0 \\ -F & F+H & -H & 0 & 0 & 0 \\ -G & -H & G+H & 0 & 0 & 0 \\ 0 & 0 & 0 & 2L & 0 & 0 \\ 0 & 0 & 0 & 0 & 2M & 0 \\ 0 & 0 & 0 & 0 & 0 & 2N \end{bmatrix} \qquad \boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{bmatrix}$$

a) Based on the above model, derive an explicit format of the evolution law for the rate of plastic strains, i.e.

$$\dot{\boldsymbol{\epsilon}}^p = \dot{\lambda} \frac{\partial \hat{f}}{\partial \boldsymbol{\sigma}}$$

b) Assume that the internal variable is given by the plastic work, i.e.

$$\dot{\kappa} = \boldsymbol{\sigma}^T \dot{\boldsymbol{\epsilon}}^p$$

Derive the format $\dot{\kappa} = \dot{\lambda} k$ and identify k.

c) Consider uniaxial loading and assume $\boldsymbol{P}$ and σ_{yo} to be known, outline a method that can be used for identification of $K(\kappa)$. Provided necessary relations that are needed in the calibration.

PROBLEM 5 (10 p.)

Assume that the initial yielding is determined by

$$F(J_2, \cos(3\theta)) = 0$$

where

$$\cos(3\theta) = \frac{3\sqrt{3}}{2}\frac{J_3}{J_2^{3/2}}$$
$$J_2 = \frac{1}{2}s_{ij}s_{ji}$$
$$J_3 = \frac{1}{3}s_{ij}s_{jk}s_{ki}$$

a) For metals and steel it is assumed that initial yield criterion is fulfilled for both σ_{ij} and $-\sigma_{ij}$. Show that this leads to

$$F(J_2, \cos(3\theta)) = F(J_2, -\cos(3\theta))$$

b) Show that for this case symmetry about $\theta = 30^o$ exists.

Hint:Use that θ and $\theta - 180^o$ fulfills the initial yield criterion together with the substitution $\theta = 30^o + \psi$, i.e. both $\theta = 30^o + \psi$ and $\theta = 30^o + \psi - 180^o = -150^o + \psi$., in $\cos(3\theta)$. Note $\psi = 0^o$ is the symmetry axis. As well as that

$$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$
$$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$$

PROBLEM 6 (10 p.)

For transversely isotropic material, such as columnar grained ice, there exists a symmetry plane of isotropy. Thus for a rotation in this plane the constitutive relation should not change.

The transformation matrix for a rotation in the x_1-x_2 plane can be written as

$$[A_{ij}] = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Moreover, the transformation of second-order tensors, i.e. stresses and strains are given by

$$\sigma'_{ij} = A_{ik}\sigma_{kl}A_{jl} \qquad \epsilon'_{ij} = A_{ik}\epsilon_{kl}A_{jl}$$

a) Show that the following hold

$$\sigma'_{12} = \frac{1}{2}\sin(2\alpha)(-\sigma_{11} + \sigma_{22}) + \cos(2\alpha)\sigma_{12}$$
$$\epsilon'_{12} = \frac{1}{2}\sin(2\alpha)(-\epsilon_{11} + \epsilon_{22}) + \cos(2\alpha)\epsilon_{12}$$

b) Given that for orthotropy the stress strain relation can be written as

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{13} & 0 & 0 & 0 \\ D_{12} & D_{22} & D_{23} & 0 & 0 & 0 \\ D_{13} & D_{23} & D_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & D_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & D_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & D_{66} \end{bmatrix} \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{12} \\ 2\epsilon_{13} \\ 2\epsilon_{23} \end{bmatrix} \tag{1}$$

Use the transformation above and consider

$$\sigma'_{12} = D_{44} 2\epsilon'_{12} \qquad \sigma'_{13} = D_{55} 2\epsilon'_{13}$$

Derive the constraints equations found by using the arguments about elastic symmetry planes.

Hint: Use that the derived expression should hold for all angles.

c) In conjunction with (1) find the relations between the components of $\boldsymbol{D}$. How many independent material parameters are resent?