MATRIX NOTATION

$$\boldsymbol{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \qquad \boldsymbol{a}^T = [a_1 \; a_2 \; a_3]$$

scalar product

$$\boldsymbol{a}^T\boldsymbol{b} = [a_1 \; a_2 \; a_3] \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

$|\boldsymbol{a}| = (\boldsymbol{a}^T\boldsymbol{a})^{1/2}$ = length of $\boldsymbol{a}$

$$\boldsymbol{B} = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix}$$

symmetric matrix if $\boldsymbol{B}^T = \boldsymbol{B}$

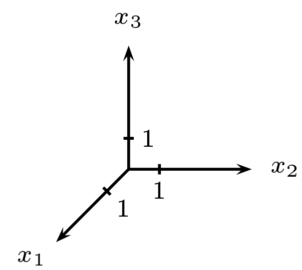
Unit matrix, for instance $\boldsymbol{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Inverse of $\boldsymbol{B} \rightarrow \boldsymbol{B}^{-1}$ (requires $\det\boldsymbol{B} \neq 0$)

$$\boldsymbol{B}^{-1}\boldsymbol{B} = \boldsymbol{I} \qquad \boldsymbol{B}\boldsymbol{B}^{-1} = \boldsymbol{I}$$

$$(\boldsymbol{AB})^T = \boldsymbol{B}^T\boldsymbol{A}^T \qquad (\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$$

CARTESIAN COORDINATE SYSTEM



- Orthogonal
- Rectangular (coordinate axes are straight lines)
- Unit length = unit along all coordinate axes

⇒

Cartesian coordinate system

(Descartes, 1596-1650)

INDEX NOTATION

- Index notation advantageous when deriving theory
- Matrix notation advantageous when it comes to numerical applications (FE-method, for instance)

$$x, y, x \quad \rightarrow \quad x_1, x_2, x_3 \quad \rightarrow \quad x_i \quad i = 1, 2, 3$$

Relation with the matrix notation

$$[x_i] = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad [a_i] = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

Rules for indices

An index can be: { free, i.e. appearing only once; dummy, i.e. it is repeated }

Summation convention:

dummy index (i.e. repeated index)
$\Rightarrow$ summation over this index

KRONECKER'S DELTA

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

$$\Rightarrow \quad [\delta_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{i.e.} \quad [\delta_{ij}] = \boldsymbol{I}$$

$$B_{ij}\delta_{jk} = B_{i1}\delta_{1k} + B_{i2}\delta_{2k} + B_{i3}\delta_{3k}$$

(summation convention)

Choose

$$i = k = 1 \quad \Rightarrow \quad B_{11}\delta_{11} + B_{12}\not{\delta}_{21} + B_{13}\not{\delta}_{31} = B_{11}$$

$$i = 1, k = 2 \quad \Rightarrow \quad B_{11}\not{\delta}_{12} + B_{12}\delta_{22} + B_{13}\not{\delta}_{32} = B_{12}$$

$$i = 1, k = 3 \quad \Rightarrow \quad B_{11}\not{\delta}_{13} + B_{12}\not{\delta}_{23} + B_{13}\delta_{33} = B_{13}$$

$$\vdots$$

$\Rightarrow$

$B_{ij}\delta_{jk} = B_{ik}$
(δ_{jk} is zero unless $i = k$)

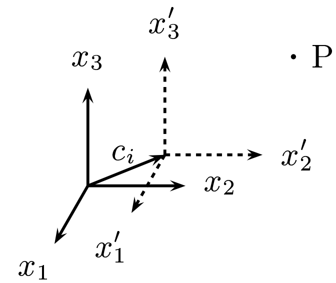
TENSORS

A tensor is a quantity that changes in a specific (simple) manner when the coordinate system is changed

Only Cartesian coordinate systems
$\Rightarrow$ only Cartesian tensors

COORDINATE TRANSFORMATIONS

Translation

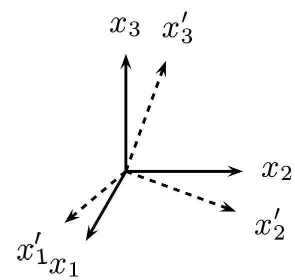


Coordinates x_i and x_i' of point P

$x_i = x_i' + c_i$

$\Rightarrow$ $x_i' = x_i - c_i$

Rotation

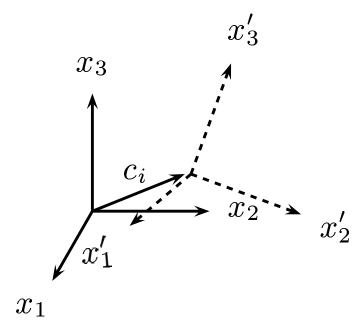


Coordinates of point P

$$x_1' = A_{11}x_1 + A_{12}x_2 + A_{13}x_3$$
$$x_2' = A_{21}x_1 + A_{22}x_2 + A_{23}x_3$$
$$x_3' = A_{31}x_1 + A_{32}x_2 + A_{33}x_3$$

$\Rightarrow$ $x_i' = \underbrace{A_{ij}}\, x_j$
transformation matrix

Translation and rotation

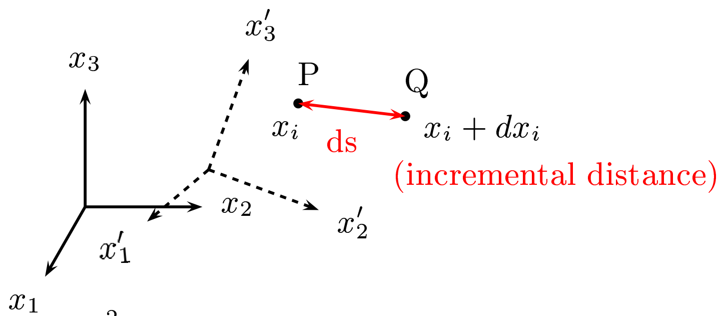


$\Rightarrow$

$x_i' = A_{ij}(x_j - c_j)$

PROPERTIES OF TRANSFORMATION MATRIX

Translation and rotation $x'_i = A_{ij}(x_j - c_j)$



$$ds^2 = dx_i dx_i = dx_j dx_j = \delta_{ij} dx_i dx_j \qquad (1)$$

Assumption: length is an invariant (i.e. independent on coordinate system)

$\Rightarrow ds^2 = ds'^2 = dx'_i dx'_i$

But
$$dx'_i = A_{ij} dx_j$$
$$ds^2 = A_{ij} dx_j A_{ik} dx_k \qquad (2)$$

(2)-(1)
$$(A_{ij}A_{ik} - \delta_{jk}) \underbrace{dx_j dx_k}_{\text{arbitrary}} = 0$$

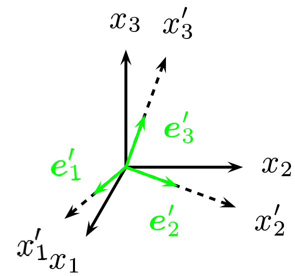
$$\Rightarrow \boxed{A_{ij}A_{ik} = \delta_{jk} \quad \text{or} \quad \boldsymbol{A}^T\boldsymbol{A} = \boldsymbol{I}}$$

$\boldsymbol{A}$ is an orthogonal matrix $\boldsymbol{A}^T = \boldsymbol{A}^{-1}$

$$\Rightarrow \boxed{\boldsymbol{A}\boldsymbol{A}^T = \boldsymbol{I} \quad \text{or} \quad A_{ik}A_{jk} = \delta_{ij}}$$

DETERMINATION OF TRANSFORMATION MATRIX

Rotation of coordinate system:



$$\boxed{\boldsymbol{x} = \boldsymbol{A}^T\boldsymbol{x}'}$$

$\boldsymbol{e}'_1$, $\boldsymbol{e}'_2$ $\boldsymbol{e}'_3$ = unit vectors along the x'_1-, x'_2- and x'_3- axis.

The components are measured in the old x_i-coordinate system

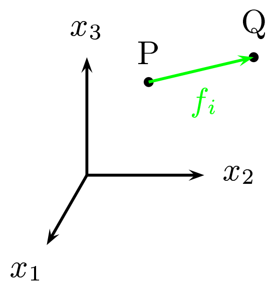
$$\boldsymbol{e}'_1 = \begin{bmatrix} e_{11} \\ e_{12} \\ e_{13} \end{bmatrix} \quad \boldsymbol{e}'_2 = \begin{bmatrix} e_{21} \\ e_{22} \\ e_{23} \end{bmatrix} \quad \boldsymbol{e}'_3 = \begin{bmatrix} e_{31} \\ e_{32} \\ e_{33} \end{bmatrix}$$

Examples: $e_{11} = \cos(x'_1 x_1)$, $e_{12} = \cos(x'_1 x_2)$

We have

$$\boldsymbol{A}^T = [\boldsymbol{e}'_1 \;\; \boldsymbol{e}'_2 \;\; \boldsymbol{e}'_3] = \begin{bmatrix} e_{11} & e_{21} & e_{31} \\ e_{12} & e_{22} & e_{32} \\ e_{13} & e_{23} & e_{33} \end{bmatrix}$$

DEFINITION OF FIRST-ORDER TENSOR



A vector has a length and direction

$\Rightarrow$ $f_i = x_i^Q - x_i^P$ is a vector

In another coordinate system

$f'_i = x_i'^Q - x_i'^P$ $\qquad$ $x'_i = A_{ij}(x_j - c_j)$

$\Rightarrow$ $f'_i = A_{ij} f_j$
Definition of first-order tensor
i.e. vector

Force = vector

Velocity of a particle $v_i = \dfrac{dx_i}{dt} = \dot{x}_i$

In new coordinate system $v'_i = \dot{x}'_i = A_{ij}\dot{x}_j$

velocity = vector
acceleration = vector

DEFINITION OF ZERO-ORDER TENSOR

Invariant = scalar = zero-order tensor = b

same value in all coordinate-systems

$\Rightarrow$ b=b'
Definition of zero-order tensor

Example = distance between two points

WHY DO TENSORS ARISE NATURALLY IN PHYSICS ?

Newton's 2 law for a particle

$$F_i = m\, a_i \qquad (1)$$

force (vector) — mass — acceleration (vector)

Assume: mass is an invariant

In another coordinate system

$$F_i' = m'\, a_i' = m\, a_i' \qquad (2)$$

(physics cannot depend on our particular choice of coordinate system)

Assume (1) - multiply by A_{ji} $\Rightarrow$

$$\underbrace{A_{ji}F_i}_{F_j'} = m\, \underbrace{A_{ji}a_i}_{a_j'}$$

$\Rightarrow$ $F_i' = m\, a_i'$ i.e. precisely (2)

If F_i and a_i were not tensors then (2) would not hold !

SECOND-ORDER TENSORS

Assume the physical relation

$$b_i = B_{ij} c_j \qquad (1)$$

where b_i and c_j are tensors (vectors)

In another coordinate system we expect

$$\underbrace{b_i'}_{A_{ik}b_k} = B_{ij}' \underbrace{c_j'}_{A_{js}c_s}$$

$$\underbrace{\underbrace{A_{it}A_{ik}}_{\delta_{tk}}\, b_k}_{b_t} = \underbrace{A_{it}B_{ij}'A_{js}}_{E_{ts}}\, c_s$$

i.e. $b_i = E_{ij}c_j$ (2)

(1)-(2) $(B_{ij} - E_{ij})\underbrace{c_j}_{\text{arbitrary}} = 0 \Rightarrow B_{ij} = E_{ij}$

$B_{ij} = A_{ki}B_{kl}'A_{lj}$ or $\boldsymbol{B} = \boldsymbol{A}^T\boldsymbol{B}'\boldsymbol{A}$
Definition of second order tensor

Kronecker's delta = second-order tensor $\delta_{ij} = \delta_{ij}'$
Isotropic tensor, same value irrespective of coordinate system

TENSORS CONT'D

If u_i = vector

then $\dfrac{\partial u_i}{\partial x_j}$ = second-order tensor

Analogously

$D_{ijkl} = A_{mi} A_{nj} D'_{mnpq} A_{pk} A_{ql}$
Definition of fourth-order tensor

Example: linear elasticity

$$\sigma_{ij} = D_{ijkl} \epsilon_{kl} \qquad \text{Hooke's law}$$

If isotropic elasticity

$$D_{ijkl} = 2G[\frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \frac{\nu}{1-2\nu}\delta_{ij}\delta_{kl}]$$

$$D_{ijkl} = D'_{ijkl}$$

i.e. isotropic fourth-order tensor.

COMMA CONVENTION - DEFINITION

Differentiation with respect to the coordinates then

$$\frac{\partial f}{\partial x_i} = f_{,i}$$

coordinate x_i

$$\frac{\partial a_i}{\partial x_j} = a_{i,j}$$

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = f_{,ij}$$

$$\frac{\partial^2 a_i}{\partial x_j \partial x_k} = a_{i,jk}$$

PROPERTIES OF TENSORS

If a tensor is zero in one coordinate system, it is zero in all coordinate systems

Example:

Define D_i by

$$D_i = \frac{\partial \sigma_{ij}}{\partial x_j} + b_i \qquad D_i = \text{tensor}$$

Equilibrium

$$D_i = 0$$

$$D'_i = A_{ij} D_j \quad \Rightarrow \quad D'_i = 0$$

If equilibrium in one coordinate system

$\Rightarrow$ we don't need to investigate equilibrium in other coordinate systems

STRAIN TENSOR

- Any displacement of a body = rigid-body motions + deformations
  Deformations = change of size and form of the body
- We expect the internal forces – i.e. the stresses – to depend on the deformation only

$\Rightarrow$

We look for a quantity – the strain tensor – that only depend on the deformations

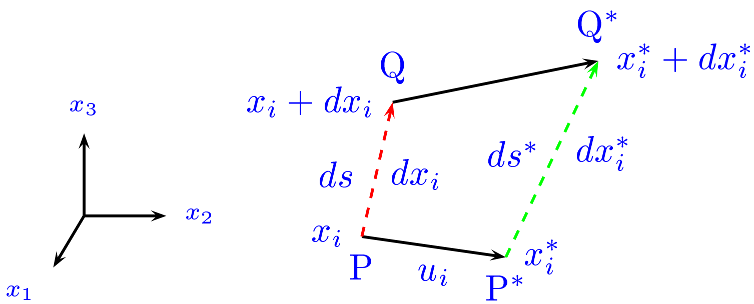
Consider a material point:

Position before def. x_i

Position after def. $\boxed{x_i^* = x_i + u_i}$

$u_i = u_i(x_k, t) =$ displacement vector

(Lagrange description)



$ds^2 = dx_k dx_k = \delta_{jk} dx_j dx_k$

$ds^{*2} = dx_i^* dx_i^* = (\delta_{jk} + u_{k,j} + u_{j,k} + u_{i,j} u_{i,k}) dx_j dx_k$

$u_{i,j} = \dfrac{\partial u_i}{\partial x_j} =$ displacement gradient

$\Rightarrow$

$$\boxed{ds^{*2} - ds^2 = (u_{k,j} + u_{j,k} + u_{i,j} u_{i,k}) dx_j dx_k}$$

We found

$$ds^{*2} - ds^2 = (u_{i,j} + u_{j,i} + u_{k,i} u_{k,j}) dx_i dx_j$$

Define

$$E_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i} + u_{k,i} u_{k,j})$$

(Green)-Lagrange's strain tensor

$$E_{ij} = E_{ji}$$

$\Rightarrow$

$$\boxed{ds^{*2} - ds^2 = 2 E_{ij} dx_i dx_j}$$

Assume

$$|u_{i,j}| << 1$$

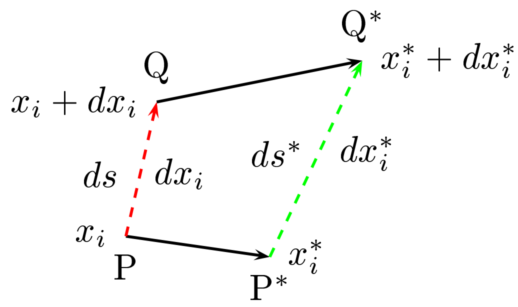
$\Rightarrow$

$$E_{ij} \approx \epsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i})$$

$$\epsilon_{ij} = \epsilon_{ji}, \qquad \epsilon_{ij} << 1$$

$\epsilon_{ij} =$ small strain tensor

PHYSICAL SIGNIFICANCE OF THE STRAIN TENSOR



$$ds^{*2} - ds^2 = 2dx_i\epsilon_{ij}dx_j$$

Define

$\boxed{n_i = \frac{dx_i}{ds}}$ = unit vector in direction dx_i

$$\Rightarrow \frac{ds^{*2} - ds^2}{2ds^2} = \frac{dx_i}{ds}\epsilon_{ij}\frac{dx_j}{ds} = n_i\epsilon_{ij}n_j$$

But

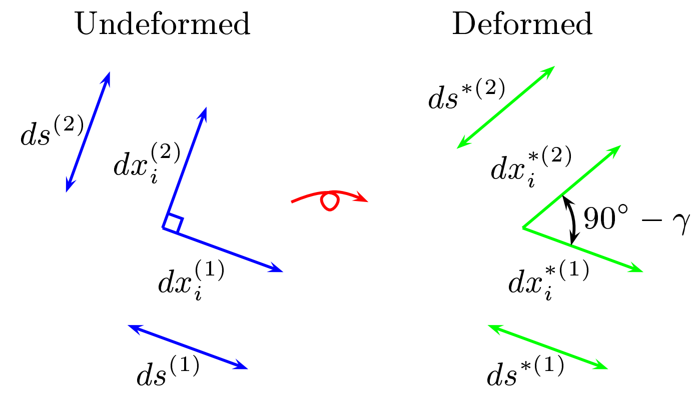
$$\frac{ds^{*2} - ds^2}{2ds^2} = \frac{(ds^* + ds)(ds^* - ds)}{2ds^2} \approx \frac{2ds(ds^* - ds)}{2ds^2} = \frac{ds^* - ds}{ds}$$

$\epsilon = \frac{ds^* - ds}{ds}$ = relative elongation

= normal strain

$\Rightarrow$

$$\boxed{\epsilon = n_i\epsilon_{ij}n_j}$$



$dx_i^{(1)}dx_i^{(2)} = 0$ orthogonality

$$\underbrace{\cos(90^\circ - \gamma)}_{\sin\gamma} = \frac{dx_i^{*(1)}}{ds^{*(1)}}\frac{dx_i^{*(2)}}{ds^{*(2)}}$$

$$x_i^* = x_i + u_i \qquad \Rightarrow \qquad dx_i^* = dx_i + du_i$$

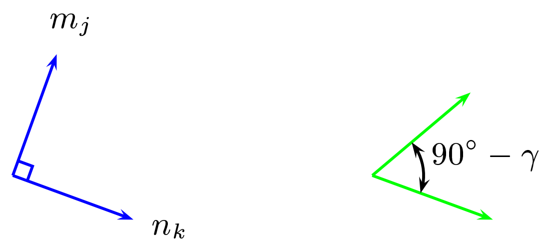
$$\sin\gamma = \underbrace{\frac{dx_k^{(1)}}{ds^{*(1)}}\frac{dx_k^{(2)}}{ds^{*(2)}}}_{=0} + (u_{k,j} + u_{j,k} + \underbrace{u_{i,j}u_{i,k}}_{\approx 0})\frac{dx_j^{*(1)}}{ds^{*(1)}}\frac{dx_k^{*(2)}}{ds^{*(2)}}$$

Moreover, $ds^{*(1)} \approx ds^{(1)}$, $ds^{*(2)} \approx ds^{(2)}$

Define unit vectors $n_i^{(1)} = \frac{dx_i^{(1)}}{ds^{(1)}}$, $n_i^{(2)} = \frac{dx_i^{(2)}}{ds^{(2)}}$

$$\boxed{\sin\gamma = 2\epsilon_{kj}n_j^{(1)}n_k^{(2)}}$$

Undeformed Deformed



$$\underbrace{\sin\gamma}_{\approx\gamma} = 2\epsilon_{kj} \underbrace{n_j^{(1)}}_{n_j} \underbrace{n_k^{(2)}}_{m_k}$$

$$\boxed{\gamma = 2n_i\epsilon_{ij}m_j}$$

$$\boxed{\epsilon_{nm} = n_i\epsilon_{ij}m_j}$$

From

$$\boxed{\begin{aligned}\epsilon_{nm} &= n_i\epsilon_{ij}m_j\\ \epsilon_{nn} &= n_i\epsilon_{ij}n_j\end{aligned}}$$

where $n_im_i = 0$, follow

$$\boxed{\text{Mohr's circle's of strain}}$$

CHANGE OF COORDINATE SYSTEM

$$x_i' = A_{ij}(x_j - c_j)$$

new coordinate — old coordinate

transformation matrix

Strain tensor

$$\epsilon_{ij}' = A_{ik}\epsilon_{kl}A_{jl} \qquad \text{or} \qquad \boldsymbol{\epsilon}' = \boldsymbol{A}\boldsymbol{\epsilon}\boldsymbol{A}^T$$

new components — old components

$$\epsilon_{ij} = A_{ki}\epsilon_{kl}'A_{lj} \qquad \text{or} \qquad \boldsymbol{\epsilon} = \boldsymbol{A}^T\boldsymbol{\epsilon}'\boldsymbol{A}$$

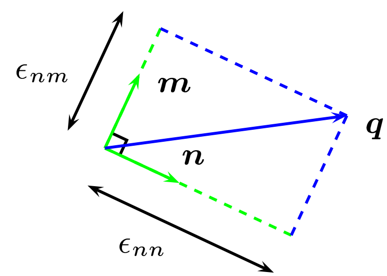
PRINCIPAL STRAINS
PRINCIPAL DIRECTIONS

$$\epsilon_{nn} = n_i\epsilon_{ij}n_j = \boldsymbol{n}^T\boldsymbol{\epsilon}\boldsymbol{n}$$

$$\epsilon_{nm} = m_i\epsilon_{ij}n_j = \boldsymbol{m}^T\boldsymbol{\epsilon}\boldsymbol{n}$$

Define $q_i = \epsilon_{ij}n_j$ i.e. $\boldsymbol{q} = \boldsymbol{\epsilon}\boldsymbol{n}$ = vector

$\Rightarrow \quad \epsilon_{nn} = \boldsymbol{n}^T\boldsymbol{q} \quad , \epsilon_{nm} = \boldsymbol{m}^T\boldsymbol{q}$



Suppose that $\epsilon_{nm} = 0 \quad \Rightarrow \quad \boldsymbol{q} = \lambda\boldsymbol{n}$

Eigenvalue problem

$$\boldsymbol{\epsilon}\boldsymbol{n} = \lambda\boldsymbol{n} \qquad \text{or} \qquad \epsilon_{ij}n_j = \lambda n_i$$

eigenvector — eigenvalue

$(\boldsymbol{\epsilon} - \lambda\boldsymbol{I})\boldsymbol{n} = 0 \quad \Rightarrow \quad \det(\boldsymbol{\epsilon} - \lambda\boldsymbol{I}) = 0$

Characteristic equation

$\Rightarrow \quad -\lambda^3 + \theta_1\lambda^2 - \theta_2\lambda + \theta_3 = 0$

PRINCIPAL STRAINS
PRINCIPAL DIRECTIONS, CONT'D

We found (characteristic equation)

$$-\lambda^3 + \theta_1 \lambda^2 - \theta_2 \lambda + \theta_3 = 0$$

where

$$\theta_1 = \epsilon_{ii}$$

$$\theta_2 = \tfrac{1}{2}\theta_1^2 - \tfrac{1}{2}\epsilon_{ij}\epsilon_{ij}$$

$$\theta_3 = \det(\epsilon_{ij})$$

In another coordinate system x_i'

$$\theta_1' = \epsilon_{ii}'$$

$$\epsilon_{ij}' = A_{ik}\epsilon_{kl}A_{jl} \quad \Rightarrow \quad \epsilon_{ii}' = \underbrace{A_{ik}\epsilon_{kl}A_{il}}_{\delta_{kl}} = \epsilon_{kk}$$

$\Rightarrow \quad \theta_1' = \theta_1$ invariant - independent on coordinate system

Likewise θ_2 and θ_3 are invariants

$\theta_1,\ \theta_2,\ \theta_3$ = strain invariants (Cauchy invariants)

the λ-values = the principal strain are invariants

PRINCIPAL STRAINS
PRINCIPAL DIRECTIONS, CONT'D

We have

$$\epsilon_{nn} = n_i \overbrace{\epsilon_{ij} n_j}^{\lambda n_i} \qquad \text{(normal strain)}$$

$$\epsilon_{mn} = m_i \underbrace{\epsilon_{ij} n_j}_{\lambda n_i} \qquad (\text{shear strain, } m_i n_i = 0)$$

$$\left.\begin{aligned} \epsilon_{nn} &= \lambda\, n_i n_i = \lambda \\ \epsilon_{mn} &= \lambda\, m_i n_i = 0 \end{aligned}\right\} \text{physical interpretation}$$

Since ϵ_{ij} is symmetric

- $\lambda_1,\ \lambda_2,\ \lambda_3$ are real (obvious!)
- The three eigenvectors n_i (the principal directions) are orthogonal

PRINCIPAL STRAINS
PRINCIPAL DIRECTIONS, CONT'D

Since

$$\left.\begin{aligned} \epsilon_{nn} &= \lambda \\ \epsilon_{mn} &= 0 \end{aligned}\right\} \quad n_i = \text{principal direct.}$$

and

the principal directions are orthogonal

⇒

We can choose a coordinate system x'_i with the axes in principal directions

In this coordinate system

$$[\epsilon'_{ij}] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

PRINCIPAL STRAINS
PRINCIPAL DIRECTIONS, CONT'D

The characteristic equation

$$-\lambda^3 + \theta_1\lambda^2 - \theta_2\lambda + \theta_3 = 0$$

where

$$\left.\begin{aligned} \theta_1 &= \epsilon_{ii} \\ \theta_2 &= \tfrac{1}{2}\theta_1 - \tfrac{1}{2}\epsilon_{ij}\epsilon_{ij} \\ \theta_3 &= \det(\epsilon_{ij}) \end{aligned}\right\} \quad \text{Cauchy invariants}$$

Other possibilities

$$\left.\begin{aligned} \tilde{I}_1 &= \epsilon_{ii} \\ \tilde{I}_2 &= \tfrac{1}{2}\epsilon_{ij}\epsilon_{ij} \\ \tilde{I}_3 &= \tfrac{1}{3}\epsilon_{ij}\epsilon_{jk}\epsilon_{ki} \end{aligned}\right\} \quad \text{"generic" invariants}$$

Their relation

$$\tilde{I}_1 = \theta_1$$

$$\tilde{I}_2 = \tfrac{1}{2}\theta_1^2 - \theta_2$$

$$\tilde{I}_3 = \tfrac{1}{3}\theta_1^3 - \theta_1\theta_2 + \theta_3$$

CAYLEY-HAMILTON'S THEOREM

First some definitions

$$\boldsymbol{\epsilon}^2 = \boldsymbol{\epsilon}\boldsymbol{\epsilon} = \epsilon_{ik}\epsilon_{kj}$$

If $\det(\boldsymbol{\epsilon}) \neq 0$ then

$$\boldsymbol{\epsilon}^{-2} = \boldsymbol{\epsilon}^{-1}\boldsymbol{\epsilon}^{-1}$$

$$\boldsymbol{\epsilon}^0 = \boldsymbol{I} \qquad (x^0 = 1)$$

From the eigenvalue problem we have

$$\boldsymbol{\epsilon}\boldsymbol{n} = \lambda\boldsymbol{n}$$

$$\boldsymbol{\epsilon}^2\boldsymbol{n} = \lambda^2\boldsymbol{n} \qquad \text{i.e.} \quad \boldsymbol{\epsilon}^2 = \lambda \underbrace{\boldsymbol{\epsilon}\boldsymbol{n}}_{\lambda\boldsymbol{n}}$$

$$\vdots$$

$$\boxed{\boldsymbol{\epsilon}^\alpha\boldsymbol{n} = \lambda^\alpha\boldsymbol{n}} \qquad \alpha = 0, \pm1, \pm2\ldots$$

CAYLEY-HAMILTON'S THEOREM, CONT'D

We found

$$\boldsymbol{\epsilon}^\alpha\boldsymbol{n} = \lambda^\alpha\boldsymbol{n}$$

and (characteristic equation)

$$-\lambda^3 + \theta_1\lambda^2 - \theta_2\lambda + \theta_3 = 0$$

Multiply with $\boldsymbol{n}$ $\Rightarrow$

$$-\lambda^3\boldsymbol{n} + \theta_1\lambda^2\boldsymbol{n} - \theta_2\lambda\boldsymbol{n} + \theta_3\boldsymbol{n} = \boldsymbol{0}$$

i.e.

$$-\boldsymbol{\epsilon}^3\boldsymbol{n} + \theta_1\boldsymbol{\epsilon}^2\boldsymbol{n} - \theta_2\boldsymbol{\epsilon}\boldsymbol{n} + \theta_3\boldsymbol{n} = \boldsymbol{0}$$

or

$$(-\boldsymbol{\epsilon}^3 + \theta_1\boldsymbol{\epsilon}^2 - \theta_2\boldsymbol{\epsilon} + \theta_3\boldsymbol{I})\boldsymbol{n} = \boldsymbol{0}$$

This equations holds for the three eigenvectors $\boldsymbol{n}_1$, $\boldsymbol{n}_2$, $\boldsymbol{n}_3$, i.e.

$$(-\boldsymbol{\epsilon}^3 + \theta_1\boldsymbol{\epsilon}^2 - \theta_2\boldsymbol{\epsilon} + \theta_3\boldsymbol{I})\underbrace{[\boldsymbol{n}_1 \;\; \boldsymbol{n}_3 \;\; \boldsymbol{n}_3]}_{\boldsymbol{A}^T} = 0$$

Post-multiplying by $\boldsymbol{A}$ $\Rightarrow$ $\boldsymbol{A}^T\boldsymbol{A} = \boldsymbol{I}$

$$\boxed{-\boldsymbol{\epsilon}^3 + \theta_1\boldsymbol{\epsilon}^2 - \theta_2\boldsymbol{\epsilon} + \theta_3\boldsymbol{I} = \boldsymbol{0}}$$

matrix equation

CAYLEY-HAMILTON'S THEOREM, CONT'D

We had (characteristic equation)

$$-\lambda^3 + \theta_1 \lambda^2 - \theta_2 \lambda + \theta_3 = 0$$

We found

$$-\boldsymbol{\epsilon}^3 + \theta_1 \boldsymbol{\epsilon}^2 - \theta_2 \boldsymbol{\epsilon} + \theta_3 \boldsymbol{I} = \boldsymbol{0}$$

i.e. Cayley-Hamilton's therorem

"the strain matrix satisfies its own characteristic equation"

generalization

$$\boldsymbol{\epsilon}^{3+\alpha} = \theta_1 \boldsymbol{\epsilon}^{2+\alpha} - \theta_2 \boldsymbol{\epsilon}^{1+\alpha} + \theta_3 \boldsymbol{\epsilon}^{\alpha}$$

DEVIATORIC STRAIN TENSOR

Define

$$e_{ij} = \epsilon_{ij} - \tfrac{1}{3}\delta_{ij}\epsilon_{kk}$$

e_{ij} = deviatoric strain tensor

It then follow that

e_{ij} and ϵ_{ij} have identical principal directions

$$e_{ii} = 0$$

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