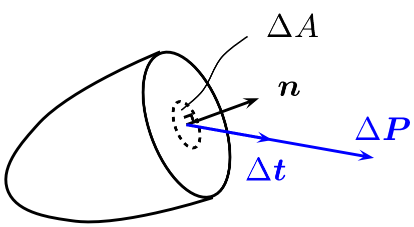
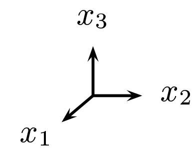
TRACTION VECTOR



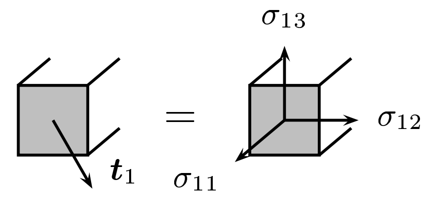
Let us define

$$\boxed{t_i = \lim_{\Delta A \to 0} \frac{\Delta P_i}{\Delta A} \quad \text{traction vector}}$$

STRESS COMPONENTS



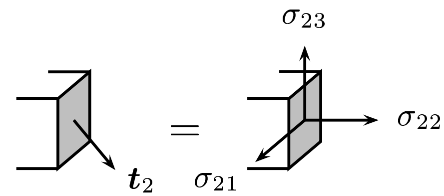
Choose $\boldsymbol{n}^T = [1 \ \ 0 \ \ 0] \ \Rightarrow \ \boldsymbol{t} = \boldsymbol{t}_1$



components of $\boldsymbol{t}_1$

$\boldsymbol{t}_1^T = [\sigma_{11} \ \ \sigma_{12} \ \ \sigma_{13}]$

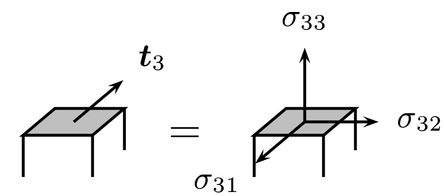
Choose $\boldsymbol{n} = [0 \ \ 1 \ \ 0] \ \Rightarrow \ \boldsymbol{t} = \boldsymbol{t}_2$



components of $\boldsymbol{t}_2$

$\boldsymbol{t}_2^T = [\sigma_{21} \ \ \sigma_{22} \ \ \sigma_{23}]$

Choose $\boldsymbol{n} = [0 \ \ 0 \ \ 1] \ \Rightarrow \ \boldsymbol{t} = \boldsymbol{t}_3$



components of $\boldsymbol{t}_3$

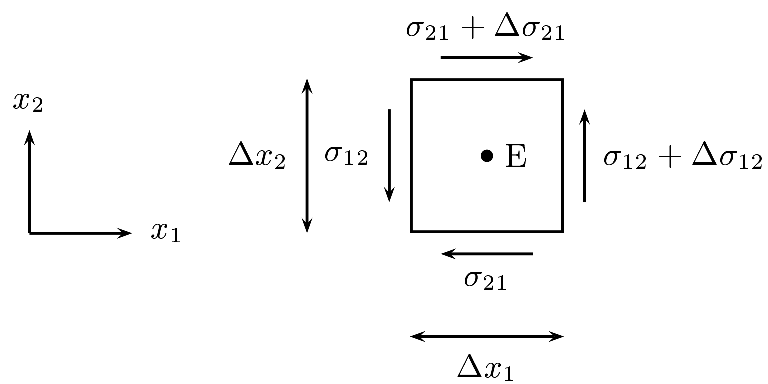
$\boldsymbol{t}_3^T = [\sigma_{31} \ \ \sigma_{32} \ \ \sigma_{33}]$

Define stress tensor

$$\boxed{\sigma_{ij} = \begin{bmatrix} \boldsymbol{t}_1^T \\ \boldsymbol{t}_2^T \\ \boldsymbol{t}_3^T \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}}$$

PROPERTIES OF σ_{ij}

Moment equilibrium



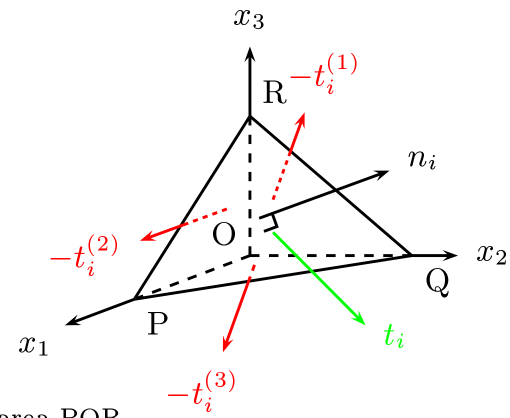
Moment about E

$$\sigma_{12} = \sigma_{21}$$

In general

$$\boxed{\begin{array}{l}\sigma_{ij} = \sigma_{ji} \\ \text{symmetric}\end{array}}$$

PROPERTIES OF σ_{ij}



dS = area PQR

dS_1 = area ROQ

dS_2 = area POR

dS_3 = area POQ

Equilibrium

$$\boldsymbol{t}dS - \boldsymbol{t}_1 \underbrace{dS_1}_{=n_1 dS} - \boldsymbol{t}_2 \underbrace{dS_2}_{=n_2 dS} - \boldsymbol{t}_3 \underbrace{dS_3}_{=n_3 dS} + \boldsymbol{b}dV = 0$$

$$\Rightarrow \quad \boldsymbol{t} \underbrace{- \boldsymbol{t}_1 n_1 - \boldsymbol{t}_2 n_2 - \boldsymbol{t}_3 n_3}_{-\boldsymbol{\sigma}^T \boldsymbol{n}} + \boldsymbol{b} \underbrace{\frac{dV}{dS}}_{\to 0} = 0$$

$$\boxed{\begin{array}{ll}t_i = \sigma_{ij} n_j & \text{Cauchy's formula} \\ \text{exactly what we wanted!} & \end{array}}$$

PROPERTIES OF σ_{ij}

We found

$$t_i = \sigma_{ij} n_j$$

In another coordinate system

$$t'_i = \sigma'_{ij} n'_j$$

t_i and n_i are vectors (first-order tensors)

Use transformation rules for t_i and n_i

$\Rightarrow$

$$\boxed{\begin{array}{lll} \sigma'_{ij} = A_{ik}\sigma_{kl}A_{jl} & \text{or} & \boldsymbol{\sigma}' = \boldsymbol{A}\boldsymbol{\sigma}\boldsymbol{A}^T \\ \sigma_{ij} = A_{ki}\sigma'_{kl}A_{lj} & \text{or} & \boldsymbol{\sigma} = \boldsymbol{A}^T\boldsymbol{\sigma}'\boldsymbol{A} \end{array}}$$

$$\boxed{\sigma_{ij} = \text{second-order tensor}}$$

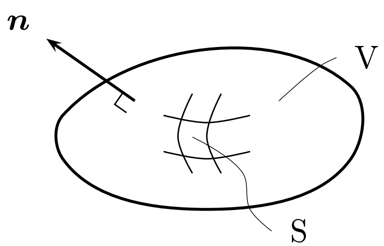
PROPERTIES OF σ_{ij}

In analogy with the strain tensor we obtain

- Mohr's circle's of stress
- Principal stress - principal directions
- Invariants
- Cayley-Hamilton's therorem
- Deviatoric stress tensor

$$\boxed{\begin{array}{l} s_{ij} = \sigma_{ij} - \frac{1}{3}\delta_{ij}\sigma_{kk} \\ s_{kk} = 0 \end{array}}$$

DIVERGENCE THEROREM OF GAUSS



n = outer unit normal vector
c = arbitrary vector; $\quad \boldsymbol{c}^T = [c_1 \;\; c_2 \;\; c_3]$

Divergence theorem

$$\int_S \boldsymbol{c}^T \boldsymbol{n} dS = \int_V \mathrm{div} \boldsymbol{c} dV$$

where

$$\mathrm{div}\boldsymbol{c} = \frac{\partial c_1}{\partial x_1} + \frac{\partial c_2}{\partial x_2} + \frac{\partial c_3}{\partial x_3} = c_{j,j}$$

$$\boldsymbol{c}^T \boldsymbol{n} = c_j n_j$$

$$\Rightarrow \quad \int_S c_j n_j dS = \int_V c_{j,j} dV$$

Generalization

$$\boxed{\int_S c_{ij} n_j dS = \int_V c_{ij,j} dV}$$

EQUATIONS OF MOTION

We found

$$\int_S c_{ij} n_j dS = \int_V c_{ij,j} dV$$

Newton's second law

$$\underbrace{\int_S t_i dS}_{\text{traction vector}} + \underbrace{\int_V b_i dV}_{\text{body force}} = \underbrace{\int_V \varrho \ddot{u}_i dV}_{\text{acceleration}}$$

We have

$$\int_S t_i dS = \int_S \sigma_{ij} n_j dS = \int_V \sigma_{ij,j} dV$$

$$\int_V (\sigma_{ij,j} + b_i - \varrho \ddot{u}_i) dV = 0$$

Volume V is arbitrary $\Rightarrow$

$$\boxed{\begin{array}{l} \sigma_{ij,j} + b_i = \varrho \ddot{u}_i \\ \text{equations of motion} \end{array}}$$

WEAK FORM - PRINCIPLE OF VIRTUAL WORK

Equations of motion

$$\sigma_{ij,j} + b_i = \varrho \underbrace{\ddot{u}_i}_{\text{acceleration}}$$

Multiply with v_i and integrate of volume

$$\int_V v_i(\sigma_{ij,j} + b_i - \varrho \ddot{u}_i) dV = 0$$

v_i = arbitrary weight vector

Use of divergence theorem and Cauchy's formula
$(t_i = \sigma_{ij} n_j)$ $\Rightarrow$

$$\boxed{\begin{gathered}\int_V \varrho v_i \ddot{u}_i dV + \int_V \underbrace{v_{i,j}\sigma_{ij}}_{\epsilon^v_{ij}\sigma_{ij}} dV = \int_S v_i t_i dS + \int_V v_i b_i dV \\ \epsilon^v_{ij} = \tfrac{1}{2}(v_{i,j} + v_{j,i})\end{gathered}}$$