We have established:

strain tensor

ϵ_{ij}

Holds for all bodies and materials

stress tensor

σ_{ij}

Holds for all bodies and materials

How they interact depends om the specific material

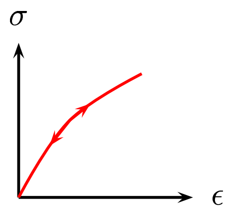
- elasticity
- plasticity
- creep
- damage

$\vdots$

⇒ constitutive mechanics

HYPER-ELASTICITY

Elastic reponse is independent on load history

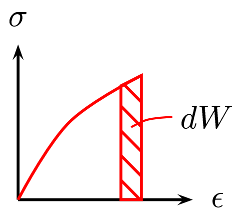


i.e $\boxed{\sigma_{ij} = \sigma_{ij}(\epsilon_{kl})}$

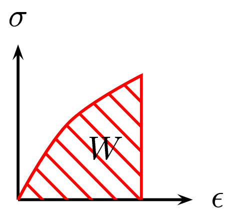
unique relation

Strain energy - uniaxial case

$\sigma = \sigma(\epsilon)$



$dW = \sigma d\epsilon$



$$W = \int_o^{\epsilon} \sigma(\epsilon^*) d\epsilon^*$$

In general

$$dW = \sigma_{ij} d\epsilon_{ij}, \qquad W = \int_o^{\epsilon_{mn}} \sigma_{ij}(\epsilon^*_{kl}) d\epsilon^*_{ij}$$

$$W = W(\epsilon_{mn}, \text{load history})$$

Assume $W = W(\epsilon_{mn})$ $\Rightarrow$ $dW = \frac{\partial W}{\partial \epsilon_{ij}} d\epsilon_{ij}$

$$(\sigma_{ij} - \frac{\partial W}{\partial \epsilon_{ij}}) d\epsilon_{ij} = 0 \qquad \Rightarrow \qquad \boxed{\sigma_{ij} = \frac{\partial W}{\partial \epsilon_{ij}}}$$

HYPER-ELASTICITY CONT'D

$$\boxed{\sigma_{ij} = \frac{\partial W}{\partial \epsilon_{ij}}}$$

From the transformation rule of second-order tensors we find $dW = \sigma_{ij} d\epsilon_{ij} = \sigma'_{ij} d\epsilon'_{ij}$

$$\boxed{W \text{ is an invariant}}$$

ISOTROPY

Strain energy should not depend on the choice of coordinate system

$$W = W(\epsilon_{ij}) = W(\tilde{I}_1, \tilde{I}_2, \tilde{I}_3)$$

where $\tilde{I}_1 = \epsilon_{ii}$, $\tilde{I}_2 = \frac{1}{2}\epsilon_{ij}\epsilon_{ij}$, $\tilde{I}_3 = \frac{1}{3}\epsilon_{ij}\epsilon_{jk}\epsilon_{ki}$

$$\sigma_{ij} = \underbrace{\frac{\partial W}{\partial \tilde{I}_1}}_{\alpha_1} \frac{\partial \tilde{I}_1}{\partial \epsilon_{ij}} + \underbrace{\frac{\partial W}{\partial \tilde{I}_2}}_{\alpha_2} \frac{\partial \tilde{I}_2}{\partial \epsilon_{ij}} + \underbrace{\frac{\partial W}{\partial \tilde{I}_3}}_{\alpha_3} \frac{\partial \tilde{I}_3}{\partial \epsilon_{ij}}$$

$$\boxed{\begin{aligned} \sigma_{ij} &= \alpha_1 \delta_{ij} + \alpha_2 \epsilon_{ij} + \alpha_3 \epsilon_{ik}\epsilon_{kj} \\ \alpha_i &= \alpha_i(\tilde{I}_1, \tilde{I}_2, \tilde{I}_3) \end{aligned}}$$

Most general hyper-elasticity for isotropic bodies!

HYPER-ELASTICITY CONT'D

We found

$$\begin{aligned} \sigma_{ij} &= \alpha_1 \delta_{ij} + \alpha_2 \epsilon_{ij} + \alpha_3 \epsilon_{ik}\epsilon_{kj} \\ \alpha_i &= \alpha_i(\tilde{I}_1, \tilde{I}_2, \tilde{I}_3) \end{aligned}$$

Choose

$$\left.\begin{aligned} \alpha_1 &= \lambda \tilde{I}_1 = \lambda \epsilon_{kk} \\ \alpha_2 &= 2\mu \\ \alpha_3 &= 0 \end{aligned}\right\} \quad \sigma_{ij} = \lambda \epsilon_{kk}\delta_{ij} + 2\mu\epsilon_{ij}$$

where

$\lambda, \mu =$ Lame's parameters

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \quad , \quad \mu = \frac{E}{2(1+\nu)} = G$$

$$\boxed{\sigma_{ij} = 2G[\epsilon_{ij} + \frac{\nu}{1-2\nu}\epsilon_{kk}\delta_{ij}]}$$

Hooke's law

HYPER-ELASTICITY CONT'D

We found

$$\boxed{\sigma_{ij} = 2G[\epsilon_{ij} + \frac{\nu}{1-2\nu}\epsilon_{kk}\delta_{ij}]}$$

(2 independent material parameters)

$$\boxed{\sigma_{kk} = 3K\epsilon_{kk}} \qquad K = \frac{E}{3(1-2\nu)}$$

$$\boxed{s_{ij} = 2Ge_{ij}} \qquad G = \frac{E}{2(1+\nu)}$$

where K and G are the bulk and shear modulus, respectively.

No coupling between volumetric and deviatoric response

HYPER-ELASTICITY CONT'D

Alternative formulation

$$\boxed{\sigma_{ij} = D_{ijkl}\epsilon_{kl}}$$

where

$$\boxed{D_{ijkl} = 2G[\frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \frac{\nu}{1-2\nu}\delta_{ij}\delta_{kl}]}$$

= fourth-order tensor

= stiffness tensor

In general $D'_{ijkl} = A_{im}A_{jn}D_{mnpq}A_{kp}A_{lq}$, but for the above

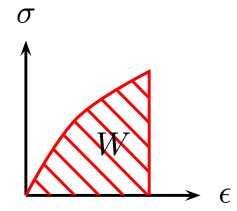
$$\boxed{D'_{ijkl} = D_{ijkl}}$$

isotropic fourth-order tensor

COMPLEMENTARY ENERGY

Strain energy $W(\epsilon_{mn}) = \int_0^{\epsilon_{mn}} \sigma_{ij}(\epsilon^*_{kl}) d\epsilon^*_{ij}$

$$\boxed{\sigma_{ij} = \frac{\partial W}{\partial \epsilon_{ij}}}$$



Legendre transformation

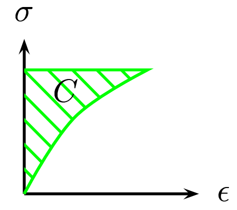
Complementary energy $C = \underbrace{\sigma_{ij}}_{\text{new variable}} \underbrace{\epsilon_{ij}}_{\text{old variable}} - W$

differentiation

$$dC = d\sigma_{ij}\epsilon_{ij} + \cancel{\sigma_{ij}} d\epsilon_{ij} - \underbrace{\cancel{\frac{\partial W}{\partial \epsilon_{ij}}}}_{\sigma_{ij}} d\epsilon_{ij} = \epsilon_{ij} d\sigma_{ij}$$

Thus

$$\boxed{\begin{aligned} &C = C(\sigma_{ij}) \\ &\epsilon_{ij} = \frac{\partial C}{\partial \sigma_{ij}} \end{aligned}}$$



COMPLEMENTARY ENERGY, CONT'D

We defined

$$C = \underbrace{\sigma_{ij}\epsilon_{ij}}_{\text{invariant}} - \underbrace{W}_{\text{invariant}}$$

$$\boxed{\begin{aligned} &C \text{ is an invariant} \\ &C = C(\sigma_{ij}) \end{aligned}}$$

ISOTROPIC NONLINEAR ELASTICITY

$$C = C(I_1, I_2, I_3) \qquad \text{or} \qquad C = C(I_1, J_2, J_3)$$

where $I_1 = \sigma_{kk}$, $J_2 = \frac{1}{2} s_{kl} s_{kl}$, $J_3 = \frac{1}{3} s_{kl} s_{lm} s_{mk}$

$$\epsilon_{ij} = \frac{\partial C}{\partial \sigma_{ij}} = \underbrace{\frac{\partial C}{\partial I_1}}_{\beta_1} \underbrace{\frac{\partial I_1}{\partial \sigma_{ij}}}_{\delta_{ij}} + \underbrace{\frac{\partial C}{\partial J_2}}_{\beta_2} \underbrace{\frac{\partial J_2}{\partial \sigma_{ij}}}_{s_{ij}} + \underbrace{\frac{\partial C}{\partial J_3}}_{\beta_3} \underbrace{\frac{\partial J_3}{\partial \sigma_{ij}}}_{\substack{s_{ik}s_{kj} \\ -\frac{2}{3}J_2\delta_{ij}}}$$

$$\boxed{\begin{aligned} &\text{Most general form} \\ &\epsilon_{ij} = \beta_1 \delta_{ij} + \beta_2 s_{ij} + \beta_3 (s_{ik}s_{kj} - \tfrac{2}{3} J_2 \delta_{ij}) \\ &\text{where} \\ &\beta_1 = \beta_1(I_1, J_2, J_3) \\ &\beta_2 = \beta_2(I_1, J_2, J_3) \\ &\beta_3 = \beta_3(I_1, J_2, J_3) \end{aligned}}$$

COMPLEMENTARY ENERGY, CONT'D

We found

$$\beta_1 = \frac{\partial C}{\partial I_1}, \qquad \beta_2 = \frac{\partial C}{\partial J_2}, \qquad \beta_3 = \frac{\partial C}{\partial J_3},$$

but from differentiation

$$\begin{aligned}
\frac{\partial \beta_1}{\partial J_2} &= \frac{\partial}{\partial J_2}\left(\frac{\partial C}{\partial I_1}\right) = \frac{\partial}{\partial I_1}\left(\frac{\partial C}{\partial J_2}\right) = \frac{\partial \beta_2}{\partial I_1} \\
\frac{\partial \beta_1}{\partial J_3} &= \frac{\partial}{\partial J_3}\left(\frac{\partial C}{\partial I_1}\right) = \frac{\partial}{\partial I_1}\left(\frac{\partial C}{\partial J_3}\right) = \frac{\partial \beta_3}{\partial I_1} \\
\frac{\partial \beta_2}{\partial J_3} &= \frac{\partial}{\partial J_3}\left(\frac{\partial C}{\partial J_2}\right) = \frac{\partial}{\partial J_2}\left(\frac{\partial C}{\partial J_3}\right) = \frac{\partial \beta_3}{\partial J_2}
\end{aligned}$$

constraints, β_1, β_2 and β_3 can not be choose independently of each other. These constraints will tell if the material can be modelled by a hyper-elasticity model.

COMPLEMENTARY ENERGY, CONT'D

We found

$$\epsilon_{ij} = \beta_1 \delta_{ij} + \beta_2 s_{ij} + \beta_3 \underbrace{(s_{ik} s_{kj} - \frac{2}{3} J_2 \delta_{ij})}_{\text{quadratic term}}$$

If Hooke format, i.e. no quadratic term

$$\Rightarrow \qquad \beta_3 = 0 \qquad \Rightarrow \qquad \begin{array}{l} \beta_1 = \beta_1(I_1, J_2) \\ \beta_2 = \beta_2(I_1, J_2) \end{array}$$

Choose $\beta_1 = \frac{\sigma_{kk}}{9K}$ and $\beta_2 = \frac{1}{2G}$

$$\epsilon_{ij} = \frac{\sigma_{kk}}{9K} \delta_{ij} + \frac{1}{2G} s_{ij}$$

Hooke format

$\sigma_{kk} = 3K\epsilon_{kk} \qquad s_{ij} = 2Ge_{ij}$

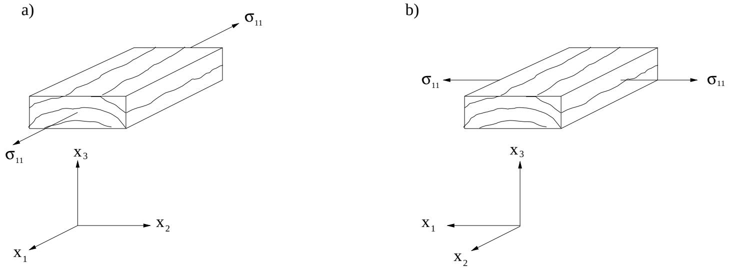
But $K = K(I_1, J_2)$ and $G = G(I_1, J_2)$

Note, now we have K and G in terms of stresses instead of strains

LINEAR HYPER-ELASTICITY - ANISOTROPY

Most general linear form $\sigma_{ij} = D_{ijkl}\epsilon_{kl}$

Piece of wood



$\sigma_{11} = E_a\epsilon_{11}$ $\qquad$ $\sigma_{11} = E_b\epsilon_{11}$

$E_a \neq E_b$ ⇒ anisotropy

Material anisotropy means that the constitutive relation takes different forms depending on the Cartesian coordinate system we use

Material isotropy means that the constitutive relation remains the same irrespective of the Cartesian system we use

---

DEFINITIONS

Consider the constitutive relation $\sigma_{ij} = D_{ijkl}\epsilon_{kl}$

If $D_{ijkl} = D_{ijkl}(x_p)$ ⇒ inhomogenous material

If D_{ijkl} independent on position ⇒ homogenous material

$$\text{Anisotropy} \quad \begin{cases} \text{homogenous} \\ \text{inhomogenous} \end{cases}$$

$$\text{Isotropy} \quad \begin{cases} \text{homogenous} \\ \text{inhomogenous} \end{cases}$$

GENERAL LINEAR ELASTICITY

$$\boxed{\sigma_{ij} = \underbrace{D_{ijkl}}_{\text{constant}}\epsilon_{kl}}$$

Consider the quantity

$$d(\frac{1}{2}\epsilon_{ij}D_{ijkl}\epsilon_{kl}) = \frac{1}{2}d\epsilon_{ij}\underbrace{D_{ijkl}\epsilon_{kl}}_{\sigma_{ij}} + \frac{1}{2}\underbrace{\epsilon_{ij}D_{ijkl}}_{\sigma_{kl}}d\epsilon_{kl}$$

$$= \frac{1}{2}d\epsilon_{ij}\sigma_{ij} + \frac{1}{2}\underbrace{\sigma_{kl}d\epsilon_{kl}}_{\sigma_{ij}d\epsilon_{ij}}$$

$\Rightarrow$

$$d(\frac{1}{2}\epsilon_{ij}D_{ijkl}\epsilon_{kl}) = \sigma_{ij}d\epsilon_{ij}$$

But since $dW = \sigma_{ij}d\epsilon_{ij}$ it follows

$$dW = d(\frac{1}{2}\epsilon_{ij}D_{ijkl}\epsilon_{kl})$$

i.e

$$\boxed{W = \tfrac{1}{2}\epsilon_{ij}D_{ijkl}\epsilon_{kl} \qquad \text{or} \qquad W = \tfrac{1}{2}\epsilon_{ij}\sigma_{ij}}$$

GENERAL LINEAR ELASTICITY, CONT’D

We found

$$W = \frac{1}{2}\epsilon_{ij}D_{ijkl}\epsilon_{kl} > 0$$

when $\epsilon_{ij} \neq 0$, i.e.

$$\boxed{D_{ijkl} \text{ positive definite}}$$

ISOTROPIC ELASTICITY

$$D_{ijkl} = 2G[\frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \frac{\nu}{1-2\nu}\delta_{ij}\delta_{kl}]$$

for $W > 0$ we find

$$\boxed{E > 0 \qquad\qquad -1 < \nu < \tfrac{1}{2}}$$

LINEAR ELASTICITY

In general

$$\sigma_{ij} = D_{ijkl}\epsilon_{kl}$$

Linear elasticity ⇒ D_{ijkl} does not depend on ϵ_{ij}

$$\Rightarrow \quad \frac{\partial \sigma_{ij}}{\partial \epsilon_{kl}} = D_{ijkl}$$

Likewise $\sigma_{kl} = D_{klij}\epsilon_{ij}$

$$\Rightarrow \quad \frac{\partial \sigma_{kl}}{\partial \epsilon_{ij}} = D_{klij}$$

But

$$\frac{\partial \sigma_{ij}}{\partial \epsilon_{kl}} = \frac{\partial}{\partial \epsilon_{kl}}\left(\frac{\partial W}{\partial \epsilon_{ij}}\right) = \frac{\partial}{\partial \epsilon_{ij}}\left(\frac{\partial W}{\partial \epsilon_{kl}}\right) = \frac{\partial \sigma_{kl}}{\partial \epsilon_{ij}}$$

$$\boxed{\begin{array}{l} D_{ijkl} = D_{klij} \\ \text{major symmerty} \end{array}}$$

Since $\sigma_{ij} = \sigma_{ji} \qquad \Rightarrow$

$$\boxed{\begin{array}{l} D_{ijkl} = D_{jikl} \\ \text{minor symmetry} \end{array}}$$

MATRIX FORMULATION

We have

$$\sigma_{ij} = D_{ijkl}\epsilon_{kl}$$

expand and collect terms

$$\begin{aligned} \sigma_{ij} =& D_{ijk1}\epsilon_{k1} + D_{ijk2}\epsilon_{k2} + D_{ijk3}\epsilon_{k3} \\ =& D_{ij11}\epsilon_{11} + D_{ij12}\epsilon_{12} + D_{ij13}\epsilon_{13} \\ &+ D_{ij21}\epsilon_{21} + D_{ij22}\epsilon_{22} + D_{ij23}\epsilon_{23} \\ &+ D_{ij31}\epsilon_{31} + D_{ij32}\epsilon_{32} + D_{ij33}\epsilon_{33} \end{aligned}$$

$$\begin{aligned} \sigma_{ij} =& D_{ij11}\epsilon_{11} + D_{ij22}\epsilon_{22} + D_{ij33}\epsilon_{33} \\ &+ D_{ij21}2\epsilon_{12} + D_{ij31}2\epsilon_{13} + D_{ij32}2\epsilon_{23} \end{aligned}$$

⇒

$$\sigma_{ij} = [D_{ij11}\ D_{ij22}\ D_{ij33}\ D_{ij21}\ D_{ij31}\ D_{ij32}] \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{12} \\ 2\epsilon_{13} \\ 2\epsilon_{23} \end{bmatrix}$$

MATRIX FORMULATION, CONT'D

We have

$$\sigma_{ij} = D_{ijkl}\epsilon_{kl}$$

expand and collect terms

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{bmatrix} = \begin{bmatrix} D_{1111} & D_{1122} & D_{1133} & D_{1121} & D_{1131} & D_{1132} \\ D_{2211} & D_{2222} & D_{2233} & D_{2221} & D_{2231} & D_{2232} \\ D_{3311} & D_{3322} & D_{3333} & D_{3321} & D_{3331} & D_{3332} \\ D_{1211} & D_{1222} & D_{1233} & D_{1221} & D_{1231} & D_{1232} \\ D_{1311} & D_{1322} & D_{1333} & D_{1321} & D_{1331} & D_{1332} \\ D_{2311} & D_{2322} & D_{2333} & D_{2321} & D_{2331} & D_{2332} \end{bmatrix} \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{12} \\ 2\epsilon_{13} \\ 2\epsilon_{23} \end{bmatrix}$$

i.e.

$$\boldsymbol{\sigma} = \boldsymbol{D}\boldsymbol{\epsilon}$$

Note $\boldsymbol{D}$ is symmetric since, e.g. $D_{2311} = D_{1132}$.

The following relation is also valid

$$\boldsymbol{\epsilon}^T\boldsymbol{\sigma} = \epsilon_{ij}\sigma_{ij}$$

But not

$$\boldsymbol{\sigma}^T\boldsymbol{\sigma} \neq \sigma_{ij}\sigma_{ij}$$

and

$$\boldsymbol{\epsilon}^T\boldsymbol{\epsilon} \neq \epsilon_{ij}\epsilon_{ij}$$

COORDINATE CHANGE IN MATRIX FORMAT

Tensor format

$$\sigma'_{ij} = A_{ik}\sigma_{kl}A_{jl} = \underbrace{A_{ik}A_{jl}}_{L_{ijkl}}\sigma_{kl}$$

in matrix format

$$\boxed{\underbrace{\boldsymbol{\sigma}'}_{6x1} = \underbrace{\boldsymbol{L}}_{6x6}\ \underbrace{\boldsymbol{\sigma}}_{6x1}}$$

to find the transformation for $\boldsymbol{\epsilon}$ we use that $\boldsymbol{\sigma}^T\boldsymbol{\epsilon}$ is an invariant

$$\boldsymbol{\sigma}^T\boldsymbol{\epsilon} = \boldsymbol{\sigma}'^T\boldsymbol{\epsilon}' = \boldsymbol{\sigma}^T\boldsymbol{L}^T\boldsymbol{\epsilon}' \quad \Rightarrow \quad \boldsymbol{\sigma}^T(\boldsymbol{\epsilon} - \boldsymbol{L}^T\boldsymbol{\epsilon}') = 0$$

$$\boxed{\boldsymbol{\epsilon} = \boldsymbol{L}^T\boldsymbol{\epsilon}' \qquad \boldsymbol{L}^T \neq \boldsymbol{L}^{-1}}$$

Transformation of the stiffness tensor, in the old coordinate system

$$\boldsymbol{\sigma} = \boldsymbol{D}\boldsymbol{\epsilon}$$

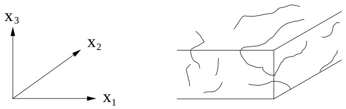
New coordinate system

$$\boldsymbol{\sigma}' = \boldsymbol{D}'\boldsymbol{\epsilon}'$$

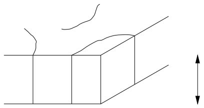
making use of the above equations

$$\boxed{\boldsymbol{D}' = \boldsymbol{L}\boldsymbol{D}\boldsymbol{L}^T}$$

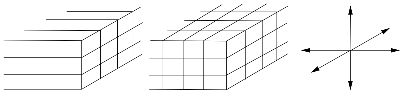
DIFFERENT FORMS OF $\boldsymbol{D}$



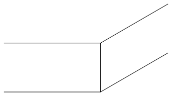
anisotropy



one symmetry plane



orthotropy
(three symmetry planes)



isotropy

$$\boxed{\boldsymbol{\sigma}=\boldsymbol{D}\boldsymbol{\epsilon}}$$

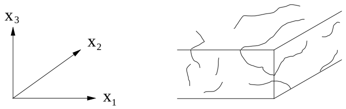
anisotropy 21 independent parameters

$$\boldsymbol{D}=\begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} & D_{15} & D_{16} \\ D_{21} & D_{22} & D_{23} & D_{24} & D_{25} & D_{26} \\ D_{31} & D_{32} & D_{33} & D_{34} & D_{35} & D_{36} \\ D_{41} & D_{42} & D_{43} & D_{44} & D_{45} & D_{46} \\ D_{51} & D_{52} & D_{53} & D_{54} & D_{55} & D_{56} \\ D_{61} & D_{63} & D_{63} & D_{64} & D_{65} & D_{66} \end{bmatrix}$$

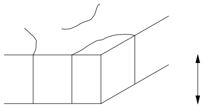
orthotropy 9 independent parameters

$$\boldsymbol{D}=\begin{bmatrix} D_{11} & D_{12} & D_{13} & 0 & 0 & 0 \\ D_{21} & D_{22} & D_{23} & 0 & 0 & 0 \\ D_{31} & D_{32} & D_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & D_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & D_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & D_{66} \end{bmatrix}$$

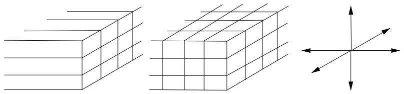
DIFFERENT FORMS OF $\boldsymbol{D}$



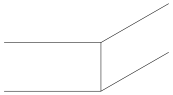
anisotropy



one symmetry plane



orthotropy
(three symmetry planes)



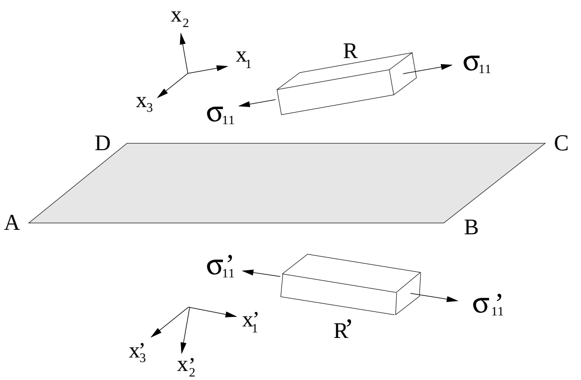
isotropy

$$\boxed{\boldsymbol{\sigma}=\boldsymbol{D}\boldsymbol{\epsilon}}$$

isotropy 2 independent parameters

$$\boldsymbol{D}=\frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}(1-2\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}(1-2\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(1-2\nu) \end{bmatrix}$$

PLANE OF ELASTIC SYMMETRY



Hooke's law $\boldsymbol{\epsilon} = \boldsymbol{C\sigma}$

take $\sigma_{11} = \sigma'_{11}$ assume this implies $\epsilon_{ij} = \epsilon'_{ij}$

more general

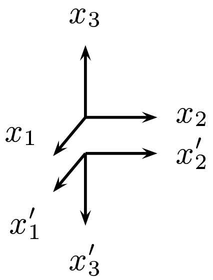
take $\sigma_{ij} = \sigma'_{ij}$ assume this implies $\epsilon_{ij} = \epsilon'_{ij}$

$\Rightarrow$ plane of elastic symmetry

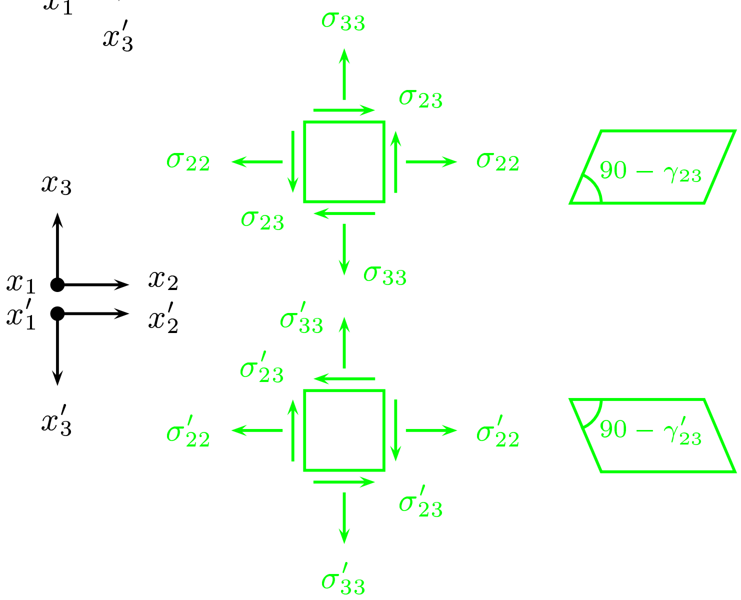
If Hooke's law takes the same form for every pair of Cartesian coordinate systems that are mirror images of each other in a certain plane, this plane is a plane of elastic symmetry

This implies that $\boldsymbol{\sigma} = \boldsymbol{D\epsilon}$ and $\boldsymbol{\sigma}' = \boldsymbol{D\epsilon}'$, i.e. same $\boldsymbol{D}$ matrix

PLANE OF ELASTIC SYMMETRY, CONT'D



$x_1 - x_2$ is assumed to be a plane of elastic symmetry



Consider a situation where the stress state $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}'$ are the same $\Rightarrow \sigma'_{23} = -\sigma_{23}$, $\sigma'_{13} = -\sigma_{13}$.

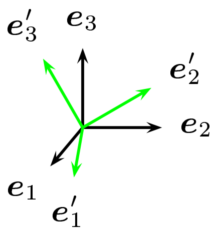
If, plane of elastic symmetry the strain state $\boldsymbol{\epsilon}$ and $\boldsymbol{\epsilon}'$ are the same $\Rightarrow \epsilon'_{23} = -\epsilon_{23}$, $\epsilon'_{13} = -\epsilon_{13}$

PLANE OF ELASTIC SYMMETRY, CONT'D

Transformation of second-order tensors

$$\underline{\boldsymbol{\sigma}}' = \boldsymbol{A}\underline{\boldsymbol{\sigma}}\boldsymbol{A}^T$$

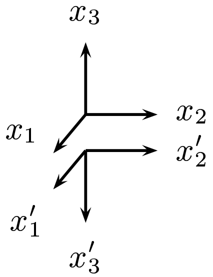
Transformation matrix



$$\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$$

$$\boldsymbol{A}^T = \begin{bmatrix} e_{11} & e_{21} & e_{31} \\ e_{12} & e_{22} & e_{32} \\ e_{13} & e_{23} & e_{33} \end{bmatrix}$$

Elastic symmetry plane $x_1 - x_2$



$$\boldsymbol{e}_1' = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \boldsymbol{e}_2' = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \boldsymbol{e}_3' = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

$$\underline{\boldsymbol{\sigma}}' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_{11} & \sigma_{12} & -\sigma_{13} \\ \sigma_{21} & \sigma_{22} & -\sigma_{23} \\ -\sigma_{31} & -\sigma_{32} & \sigma_{33} \end{bmatrix}$$

Same transformation for the strain tensor

PLANE OF ELASTIC SYMMETRY, CONT'D

we have

$$\begin{aligned} \sigma_{23}' &= -\sigma_{23} \qquad & \sigma_{13}' &= -\sigma_{13} \\ \epsilon_{23}' &= -\epsilon_{23} \qquad & \epsilon_{13}' &= -\epsilon_{13} \end{aligned}$$

otherwise $\sigma_{ij}' = \sigma_{ij}$ and $\epsilon_{ij}' = \epsilon_{ij}$ and

$$\boldsymbol{\sigma} = \boldsymbol{D}\boldsymbol{\epsilon} \qquad (1)$$

$$\boldsymbol{\sigma}' = \boldsymbol{D}\boldsymbol{\epsilon}' \qquad (2)$$

expand equations

$$(1) \quad \Rightarrow \quad \begin{aligned} \sigma_{11} = {} & D_{11}\epsilon_{11} + D_{12}\epsilon_{22} + D_{13}\epsilon_{33} \\ & + 2D_{14}\epsilon_{12} + 2D_{15}\epsilon_{13} + 2D_{16}\epsilon_{23} \end{aligned}$$

$$(2) \quad \Rightarrow \quad \begin{aligned} \sigma_{11}' = {} & D_{11}\epsilon_{11}' + D_{12}\epsilon_{22}' + D_{13}\epsilon_{33}' \\ & + 2D_{14}\epsilon_{12}' + 2D_{15}\epsilon_{13}' + 2D_{16}\epsilon_{23}' \end{aligned}$$

$$\begin{aligned} \sigma_{11} = {} & D_{11}\epsilon_{11} + D_{12}\epsilon_{22} + D_{13}\epsilon_{33} \\ & + 2D_{14}\epsilon_{12} - 2D_{15}\epsilon_{13} - 2D_{16}\epsilon_{23} \end{aligned}$$

Then $D_{15} = D_{16} = 0$

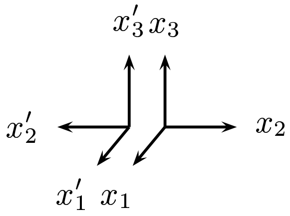
In the same manner $D_{25}=D_{26}=D_{35}=D_{45}=D_{46}=0$

One symmetry plane

$$\Rightarrow \qquad \boldsymbol{D} = \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} & 0 & 0 \\ D_{21} & D_{22} & D_{23} & D_{24} & 0 & 0 \\ D_{31} & D_{32} & D_{33} & D_{34} & 0 & 0 \\ D_{41} & D_{42} & D_{43} & D_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & D_{55} & D_{56} \\ 0 & 0 & 0 & 0 & D_{65} & D_{66} \end{bmatrix}$$

PLANE OF ELASTIC SYMMETRY, CONT'D

$x_1 - x_3$ is assumed to be a plane of elastic symmetry



$$\boldsymbol{e}'_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \boldsymbol{e}'_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \quad \boldsymbol{e}'_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Stress $\boldsymbol{\sigma}$ and strain $\boldsymbol{\epsilon}$ in the x'_i system

$$\underline{\boldsymbol{\sigma}}' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_{11} & -\sigma_{12} & \sigma_{13} \\ -\sigma_{21} & \sigma_{22} & -\sigma_{23} \\ \sigma_{31} & -\sigma_{32} & \sigma_{33} \end{bmatrix}$$

$$\underline{\boldsymbol{\epsilon}}' = \begin{bmatrix} \epsilon_{11} & -\epsilon_{12} & \epsilon_{13} \\ -\epsilon_{21} & \epsilon_{22} & -\epsilon_{23} \\ \epsilon_{31} & -\epsilon_{32} & \epsilon_{33} \end{bmatrix}$$

PLANE OF ELASTIC SYMMETRY, CONT'D

we have

$$\begin{aligned} \sigma'_{12} &= -\sigma_{12} & \sigma'_{23} &= -\sigma_{23} \\ \epsilon'_{12} &= -\epsilon_{12} & \epsilon'_{23} &= -\epsilon_{23} \end{aligned}$$

otherwise $\sigma'_{ij} = \sigma_{ij}$ and $\epsilon'_{ij} = \epsilon_{ij}$ and

$$\boldsymbol{\sigma} = \boldsymbol{D}\boldsymbol{\epsilon} \qquad (1)$$

$$\boldsymbol{\sigma}' = \boldsymbol{D}\boldsymbol{\epsilon}' \qquad (2)$$

expand equations

$$(1) \Rightarrow \quad \sigma_{11} = D_{11}\epsilon_{11} + D_{12}\epsilon_{22} + D_{13}\epsilon_{33} + 2D_{14}\epsilon_{12}$$

$$(2) \Rightarrow \quad \sigma'_{11} = D_{11}\epsilon'_{11} + D_{12}\epsilon'_{22} + D_{13}\epsilon'_{33} + 2D_{14}\epsilon'_{12}$$

$$\sigma_{11} = D_{11}\epsilon_{11} + D_{12}\epsilon_{22} + D_{13}\epsilon_{33} - 2D_{14}\epsilon_{12}$$

Then $D_{14} = 0$, in the same manner $D_{24}=D_{34}=0$

$$(1) \Rightarrow \quad \sigma_{13} = 2D_{55}\epsilon_{13} + 2D_{56}\epsilon_{23}$$

$$(2) \Rightarrow \quad \sigma'_{13} = 2D_{55}\epsilon'_{13} + 2D_{56}\epsilon'_{23}$$

$$\sigma_{11} = 2D_{55}\epsilon_{13} - 2D_{56}\epsilon_{23}$$

Then $D_{56} = 0$

Two symmetry planes

$$\Rightarrow \quad \boldsymbol{D} = \begin{bmatrix} D_{11} & D_{12} & D_{13} & 0 & 0 & 0 \\ D_{21} & D_{22} & D_{23} & 0 & 0 & 0 \\ D_{31} & D_{32} & D_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & D_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & D_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & D_{66} \end{bmatrix}$$

PLANE OF ELASTIC SYMMETRY, CONT'D

Two symmetry planes ⇒ same as three symmetry planes

⇒ orthotropic material

$$\begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{12} \\ 2\epsilon_{13} \\ 2\epsilon_{23} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & -\frac{\nu_{31}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & -\frac{\nu_{32}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{13}}{E_1} & -\frac{\nu_{23}}{E_2} & \frac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{12}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{13}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{23}} \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{bmatrix}$$

Note the symmetry properties $\frac{\nu_{21}}{E_2} = \frac{\nu_{12}}{E_1}$, $\frac{\nu_{31}}{E_3} = \frac{\nu_{13}}{E_1}$ and $\frac{\nu_{32}}{E_3} = \frac{\nu_{23}}{E_2}$

THERMO-ELASTICITY

Split of the strain tensor

$$\epsilon_{ij} = \epsilon^e_{ij} + \epsilon^T_{ij}$$

total strains elastic strains thermal strains

Hooke's law

$$\boxed{\sigma_{ij} = D_{ijkl}\underbrace{(\epsilon_{kl} - \epsilon^T_{kl})}_{\epsilon^e_{kl}}}$$

For isotropic materials

$$\epsilon^T_{ij} = \alpha\,\Delta T\,\delta_{ij}$$

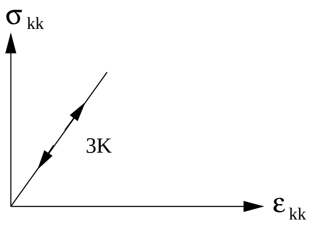
thermal expansion coefficient

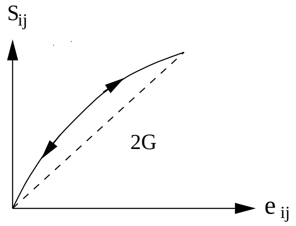
temp. change from reference temperature

NONLINEAR ISOTROPIC HOOKE FORMULATION

For metals and steel, the volumetric response is linear elastic and all nonlinearity is related to the deviatoric response

⇒ Choose $K = constant$, $G = G(J_2)$



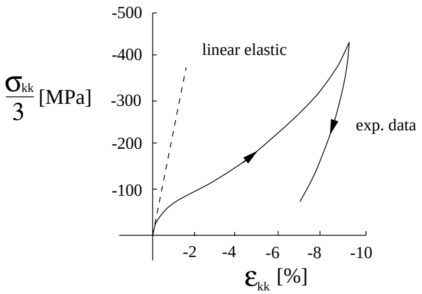


uncoupled, all nonlinearity related to deviatoric response

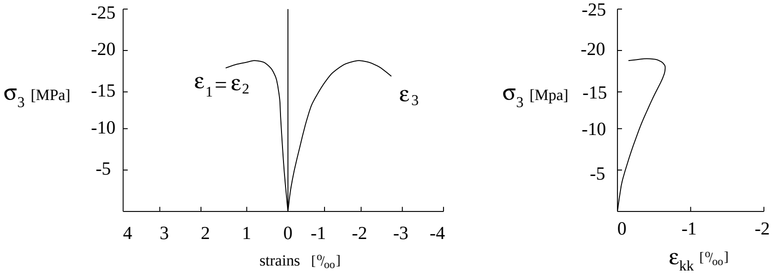
(”total” or ”deformational” plasticity)

NONLINEAR ISOTROPIC HOOKE FORMULATION

For concrete, soil and rocks, the volumetric and deviatoric response is coupled and both the volumetric and deviatoric response are nonlinear



Hydrostatic compression of concrete ($\sigma_1 = \sigma_2 = \sigma_3 < 0$)



Uniaxial compression of concrete ($\sigma_1 = \sigma_2 = 0, \sigma_3 < 0$)

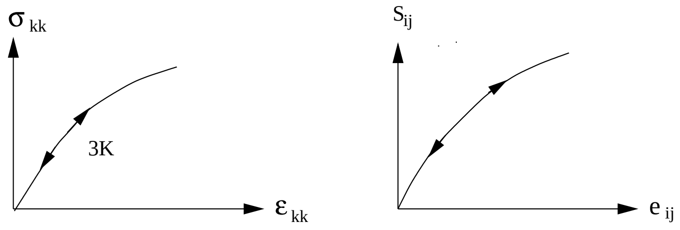
NONLINEAR ISOTROPIC HOOKE FORMULATION

Most general Hooke format

$$\sigma_{ii} = 3K\epsilon_{ii} \qquad s_{ij} = 2Ge_{ij}$$

where

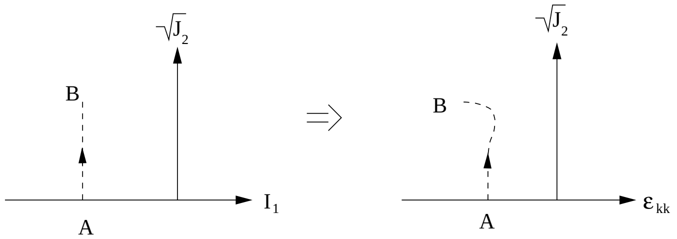
$$K = K(I_1, J_2) \qquad G = G(I_1, J_2)$$



Nonlinear both in volumetric and deviatoric response

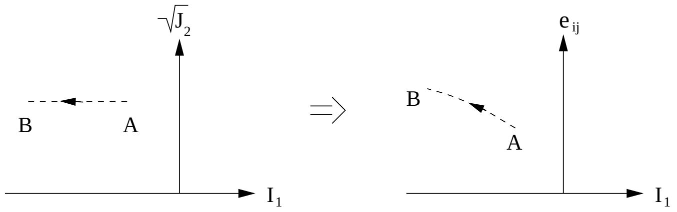
NONLINEAR ISOTROPIC HOOKE FORMULATION

We can model coupling effects!!



constant hydrost. stress - increasing dev. stress

$$\overbrace{\sigma_{ii}}^{const} = 3K(\overbrace{I_1}^{const}, \underbrace{J_2}_{varies})\underbrace{\epsilon_{ii}}_{varies}$$



constant dev. stress - change of dev. stress

$$\overbrace{s_{ij}}^{const} = 2G(\underbrace{I_1}_{varies}, \overbrace{J_2}^{const})\underbrace{e_{ij}}_{varies}$$

A simple nonlinear Hooke formulation allows for coupling effects between volumetric and deviatoric response