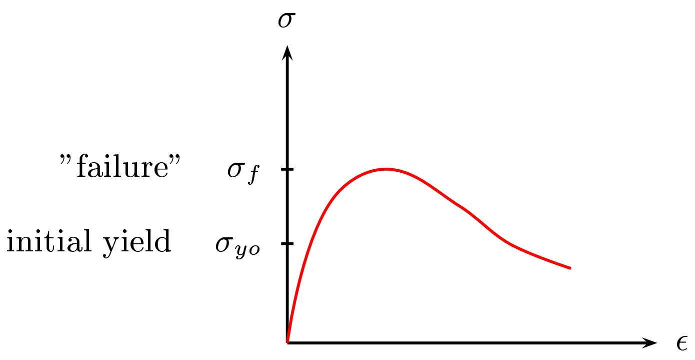
## FAILURE AND INITIAL YIELD CRITERIA




Determination of "failure" and initial yielding is (almost) trivial for uniaxial loading

What about general 3D-loading?

$$f(\sigma_{ij}) = 0$$

Coordinate system arbitrary

$\Rightarrow$ $f = 0$ in all coordinate systems

$\Rightarrow$ $f$ is an invariant – a scalar

**$f$ is an invariant**

for example: $f(\sigma_{ij}\sigma_{ij}) = 0$



## ISOTROPIC MATERIALS

In general $f(\sigma_{ij}) = 0$ or $f(\sigma_1, \sigma_2, \sigma_3, \boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{n}_3) = 0$.

Isotropic material cannot depend of the choosen coordinate system

$$f(\sigma_1, \sigma_2, \sigma_3) = 0$$

or

$$f(I_1, I_2, I_3) = 0$$

where $I_1 = \sigma_{ii}$, $I_2 = \frac{1}{2}\sigma_{ij}\sigma_{ji}$ and $I_3 = \frac{1}{3}\sigma_{ij}\sigma_{jk}\sigma_{ki}$

We want to seperate hydrostatic stress from deviatoric stresses

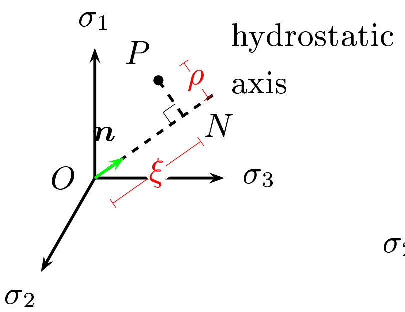
Alternatively

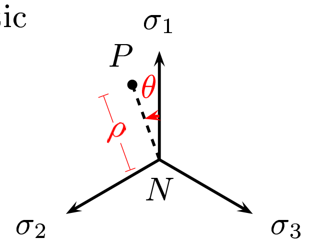
$$f(I_1, J_2, J_3) = 0$$

where $J_2 = \frac{1}{2}s_{ij}s_{ji}$ and $J_3 = \frac{1}{3}s_{ij}s_{jk}s_{ki}$

$$f(I_1, J_2, \cos 3\theta) = 0$$

## HAIGH-WESTERGAARD COORDINATE SYSTEM – GEOMETRICAL INTERPRETATION OF STRESS INVARIANTS




Deviatoric plane

Instead of coordinates $(\sigma_1, \sigma_2, \sigma_3)$ we may use coordinates $(\xi, \rho, \theta)$

$$\xi = \boldsymbol{n}^T \overline{OP} = \frac{1}{\sqrt{3}}[1\ 1\ 1] \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{bmatrix} = \frac{I_1}{\sqrt{3}}$$

$$\overline{ON} = \xi \boldsymbol{n} = \frac{I_1}{\sqrt{3}} \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{I_1}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\overline{NP} = \overline{OP} - \overline{ON} = \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{bmatrix} - \frac{I_1}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix}$$

$$\rho = \sqrt{2J_2} \qquad \cos 3\theta = \frac{3\sqrt{3}J_3}{2J_2^{3/2}}$$



## ISOTROPIC MATERIALS

$$\left.\begin{aligned} f(\sigma_1, \sigma_2, \sigma_3) &= 0 \\ f(I_1, I_2, I_3) &= 0 \\ f(\rho, \xi, \cos 3\theta) &= 0 \\ f(I_1, J_2, \cos 3\theta) &= 0 \end{aligned}\right\} \quad \text{equivalent expressions}$$

Especially convenient

$$f(I_1, J_2, \cos 3\theta) = 0$$

- hydrostatic stress
- amount of deviatoric stress
- direction of deviatoric stress

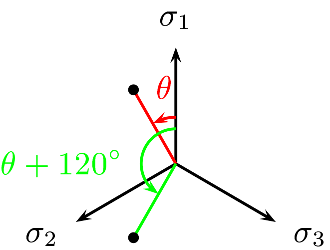
decoupling effects

$\cos 3\theta$-term ⇒ symmetry properties

## SYMMETRY PROPERTIES

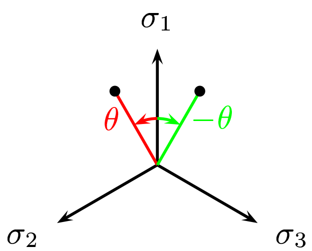
In deviatoric plane $I_1$=constant

$$f(I_1, J_2, \cos 3\theta) = 0$$




$$\cos 3\theta = \cos(3\theta + 360°) = \cos 3(\theta + 120°)$$

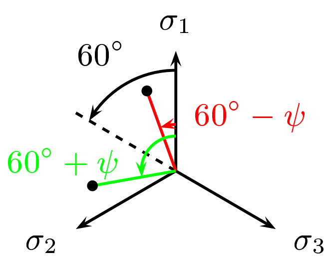
120° period




$$\cos 3\theta = \cos 3(-\theta)$$

symmerty about $\theta$=0

(symmetry about $\theta$=120° and $\theta$=240○)




$$\cos 3(60° - \psi) = \cos(180° - 3\psi) = \cos(-3\psi) = -\cos 3\psi$$

$$\cos 3(60° + \psi) = \cos(180° + 3\psi) = -\cos 3\psi$$

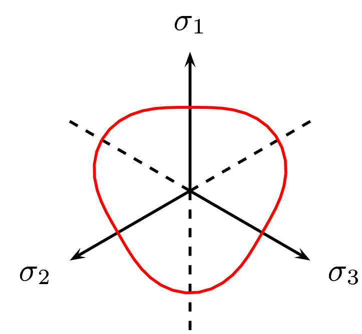
symmetry about $\theta$=60°



## SYMMETRY PROPERTIES

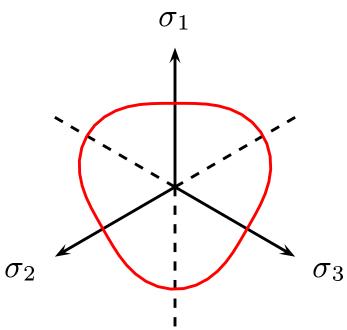
In deviatoric plane $I_1$=constant

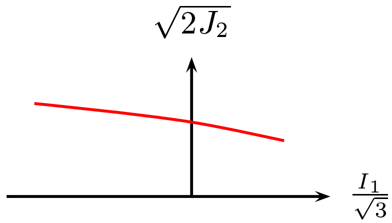
$$f(I_1, J_2, \cos 3\theta) = 0$$




Complete general properties
(+convexity; exp. fact.)

## GENERAL EXPERIMENTAL EVIDENCE



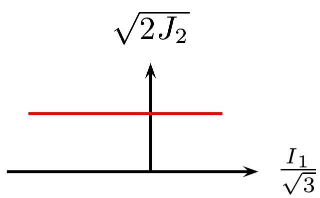




Dev. plane = plane perpendicular to the hydrostatic axis, $I_1$ constant

Meridian plane = plane containing the hydrostatic axis, $\theta$ constant

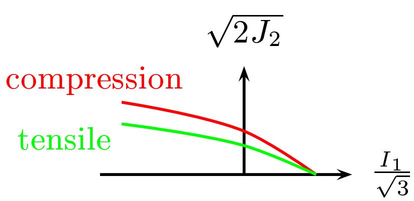
### METALS, STEEL

Yielding independent on hydro. stress $I_1$
$f(J_2, \cos 3\theta) = 0$




### CONCRETE, SOILS, ROCKS

Failure depends strongly on $I_1$
$f(I_1, J_2, \cos 3\theta) = 0$
*cos*$3\theta$ strong influence



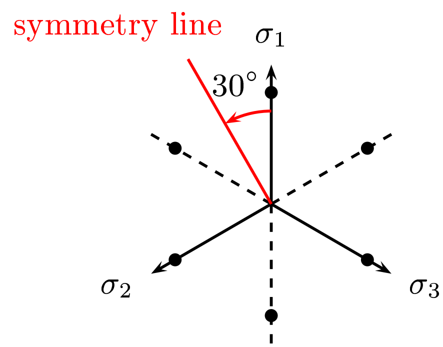


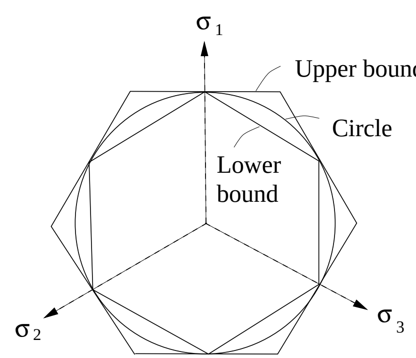

## METALS AND STEEL

No influence of $I_1$

$$f(J_2, \cos 3\theta) = 0$$

If $\sigma_{ij}$ same result as $-\sigma_{ij}$ (tension = compression), $\cos 3\theta$-term shows $30°$-symmetry

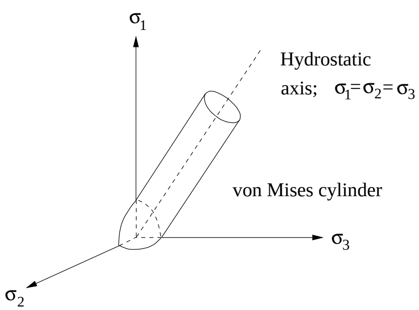


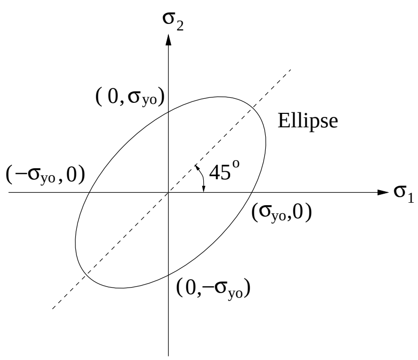




We focus on lower bound and circle

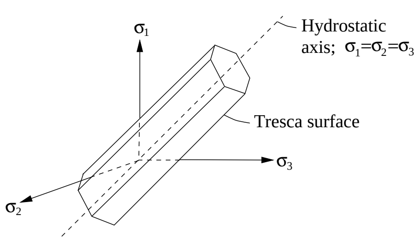
# METALS AND STEEL

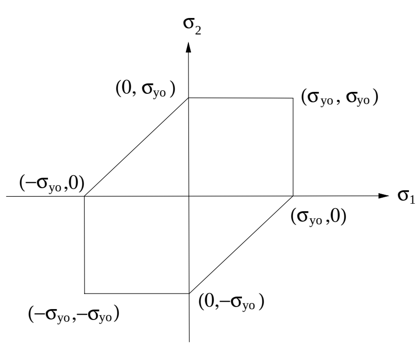
## von Mises




## Tresca

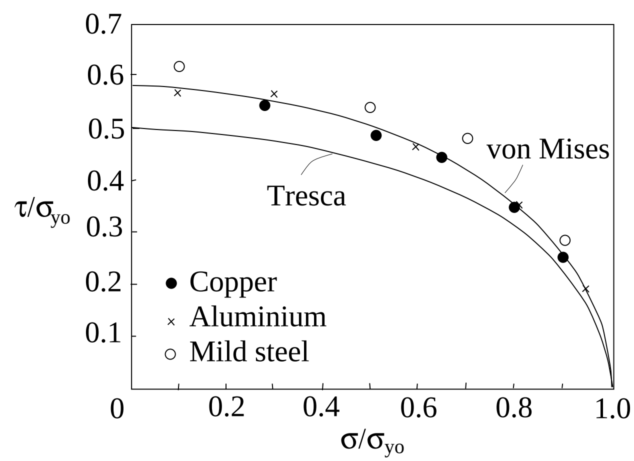


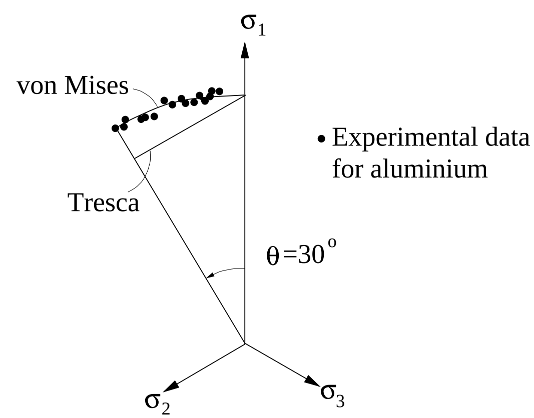



# METALS AND STEEL, EXPER.

Taylor & Quinney (1931)




Lianis & Ford (1957)



## CEMENTITIOUS MATERIALS

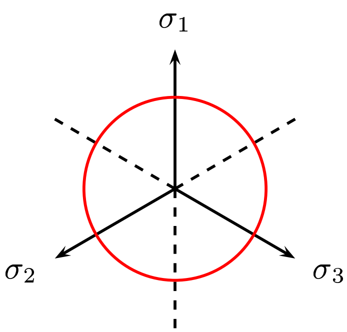
$$f(I_1, J_2, \cos 3\theta) = 0$$

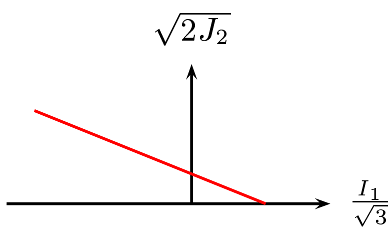
Simplest possible (Schleicher (1926))

$$f(I_1, J_2) = 0$$

Linear expression (Drucker & Prager (1952))

$$\boxed{\sqrt{J_2} + \alpha I_1 - \beta = 0}$$




Convenient mathematical expression

Use it with care

Very poor - except for small friction angles



## 4-PARAMETER CRITERION

$$A\frac{J_2}{\sigma_c} + \lambda(\theta)\frac{\sqrt{J_2}}{\sigma_c} + B\frac{I_1}{\sigma_c} - 1 = 0$$

where

$$\lambda(\theta) = \begin{cases} K_1 \cos[\frac{1}{3}\arccos(K_2 \cos 3\theta)] & \cos 3\theta \geq 0 \\ K_1 \cos[\frac{\pi}{3} - \frac{1}{3}\arccos(K_2 \cos 3\theta)] & \cos 3\theta \leq 0 \end{cases}$$

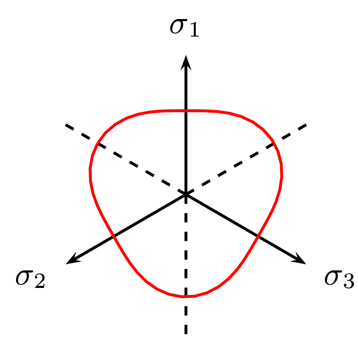
Parameters $A$, $B$, $K_1$ and $K_2$.

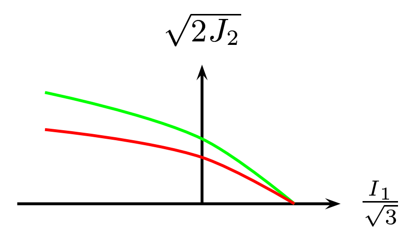
If

$$A = K_2 = 0 \quad \Rightarrow \quad \text{Drucker \& Prager}$$

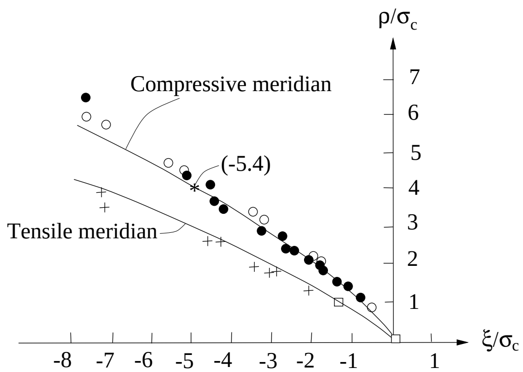
If

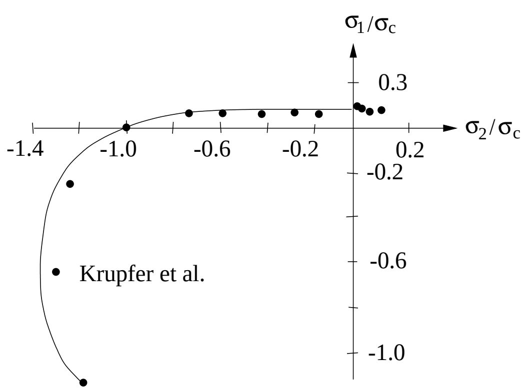
$$A = B = K_2 = 0 \quad \Rightarrow \quad \text{von Mises}$$



# 4-PARAMETER CRITERION
# –CONCRETE–






# ANISOTROPIC CRITERIA

In general $f(\sigma_{ij}) = 0$ is an invariant

Assume: initial yield independent on hydrostatic stress

Assume

$$s_{ij} P_{ijkl} s_{kl} - 1 = 0 \quad \text{or} \quad \boldsymbol{s}^T \hat{\boldsymbol{P}} \boldsymbol{s} - 1 = 0$$

where

$$\boldsymbol{s}^T = [s_{11} \;\; s_{22} \;\; s_{33} \;\; s_{12} \;\; s_{13} \;\; s_{23}]$$

Choose

$$P_{ijkl} = \frac{3}{4\sigma_{yo}^2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

in matrix format (where $s_{11} + s_{22} + s_{33} = 0$ is used)

$$\hat{\boldsymbol{P}} = \frac{1}{2\sigma_{yo}^2} \begin{bmatrix} 2 & -1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 \end{bmatrix}$$

$$\Rightarrow \quad \frac{3}{2\sigma_{yo}^2} s_{ij} s_{ij} - 1 = 0$$

i.e. von Mises criteria

## ANISOTROPIC CRITERIA

Consider (von Mises (1928))

$$\sigma_{ij}P_{ijkl}\sigma_{kl} - 1 = 0 \quad \text{or} \quad \boldsymbol{\sigma}^T\boldsymbol{P}\boldsymbol{\sigma} - 1 = 0$$

It turns out that $\boldsymbol{P} = \boldsymbol{P}^T$, i.e. 21 parameters

$$\boldsymbol{P} = \begin{bmatrix} A & -F & -G & P_{14} & P_{15} & P_{16} \\ -F & B & -H & P_{24} & P_{25} & P_{26} \\ -G & -H & C & P_{34} & P_{35} & P_{36} \\ P_{14} & P_{24} & P_{34} & 2L & P_{45} & P_{46} \\ P_{15} & P_{25} & P_{35} & P_{45} & 2M & P_{56} \\ P_{16} & P_{26} & P_{36} & P_{46} & P_{56} & 2N \end{bmatrix}$$

Assume: Orthotropy (3 symmerty planes)

$$\boldsymbol{P} = \begin{bmatrix} A & -F & -G & 0 & 0 & 0 \\ -F & B & -H & 0 & 0 & 0 \\ -G & -H & C & 0 & 0 & 0 \\ 0 & 0 & 0 & 2L & 0 & 0 \\ 0 & 0 & 0 & 0 & 2M & 0 \\ 0 & 0 & 0 & 0 & 0 & 2N \end{bmatrix}$$

9 parameters to determine.



## ANISOTROPIC CRITERIA

Assume: Yielding should only depend on deviatoric stresses, split stresses $\boldsymbol{\sigma} = \boldsymbol{s} + \boldsymbol{e}$

$$\boldsymbol{s}^T\boldsymbol{P}\boldsymbol{s} + (2\boldsymbol{s}^T + \boldsymbol{e})\boldsymbol{P}\boldsymbol{e} - 1 = 0$$

since $\boldsymbol{e}$ and $\boldsymbol{s}$ are independent

$$\boldsymbol{P}\boldsymbol{e} = \boldsymbol{0}$$

$$\boldsymbol{s}^T\boldsymbol{P}\boldsymbol{s} - 1 = 0$$

Constraint $\boldsymbol{P}\boldsymbol{e} = \boldsymbol{0}$

$$\begin{aligned} A - F - G &= 0 \\ -F + B - H &= 0 \\ -G - H + C &= 0 \end{aligned}$$

Orthotropy – only influence of deviatoric stresses

$$\boldsymbol{P} = \begin{bmatrix} F+G & -F & -G & 0 & 0 & 0 \\ -F & F+H & -H & 0 & 0 & 0 \\ -G & -H & G+H & 0 & 0 & 0 \\ 0 & 0 & 0 & 2L & 0 & 0 \\ 0 & 0 & 0 & 0 & 2M & 0 \\ 0 & 0 & 0 & 0 & 0 & 2N \end{bmatrix}$$

6 parameters, Hill (1948,1950)

## ANISOTROPIC CRITERIA
## – CALIBRATION –

We found $\boldsymbol{s}^T \boldsymbol{P} \boldsymbol{s} - 1 = 0$ or

$$F(s_{11}-s_{22})^2 + G(s_{11}-s_{33})^2 + H(s_{22}-s_{33})^2 + 2Ls_{12}^2 + 2Ms_{13}^2 + 2Ns_{23}^2 - 1 = 0$$

Need to determine 6 parameters

Uniaxial test 1-direction

$$\sigma_{11} \neq 0 \qquad \sigma_{ij} = 0 \quad \text{otherwise}$$

From $s_{ij} = \sigma_{ij} - \sigma_{kk}\delta_{ij}/3$

$$s_{11} = \sigma_{11} - \frac{1}{3}\sigma_{11}, \quad s_{22} = -\frac{1}{3}\sigma_{11}, \quad s_{33} = -\frac{1}{3}\sigma_{11}$$

$$\Rightarrow \qquad F\sigma_{11}^2 + G\sigma_{11}^2 - 1 = 0$$

Assume yielding starts at $\sigma_{11} = \sigma_{yo}^{11}$

$$F + G = \frac{1}{(\sigma_{yo}^{11})^2}$$

In the same manner, uniaxial loading in 2 and 3 directions

$$F + H = \frac{1}{(\sigma_{yo}^{22})^2} \qquad G + H = \frac{1}{(\sigma_{yo}^{33})^2}$$



## ANISOTROPIC CRITERIA
## – CALIBRATION –

We found

$$F+G = \frac{1}{(\sigma_{yo}^{11})^2} \qquad F+H = \frac{1}{(\sigma_{yo}^{22})^2} \qquad G+H = \frac{1}{(\sigma_{yo}^{33})^2}$$

Three parameters, three equations

$$F = \frac{1}{2}\left[\frac{1}{(\sigma_{yo}^{11})^2} + \frac{1}{(\sigma_{yo}^{22})^2} - \frac{1}{(\sigma_{yo}^{33})^2}\right]$$
$$G = \frac{1}{2}\left[\frac{1}{(\sigma_{yo}^{11})^2} - \frac{1}{(\sigma_{yo}^{22})^2} + \frac{1}{(\sigma_{yo}^{33})^2}\right]$$
$$H = \frac{1}{2}\left[-\frac{1}{(\sigma_{yo}^{11})^2} + \frac{1}{(\sigma_{yo}^{22})^2} + \frac{1}{(\sigma_{yo}^{33})^2}\right]$$

Next, applying shear stress $\sigma_{12}$ only

$$s_{12} = \sigma_{12} \qquad \sigma_{ij} = 0 \qquad \text{otherwise}$$

Hill's yield criterion

$$2Ls_{12}^2 - 1 = 0$$

Assume yielding starts at $s_{12} = \sigma_{12} = \sigma_{yo}^{12}$, i.e.

$$L = \frac{1}{2(\sigma_{yo}^{12})^2}$$

In the same manner for the 2 other paramaters

$$M = \frac{1}{2(\sigma_{yo}^{13})^2} \qquad N = \frac{1}{2(\sigma_{yo}^{23})^2}$$

## ANISOTROPIC CRITERIA

Hill's ortotropic yield criterion (1948,1950)

$$F(s_{11}-s_{22})^2+G(s_{11}-s_{33})^2+H(s_{22}-s_{33})^2$$
$$+2Ls_{12}^2+2Ms_{13}^2+2Ns_{23}^2-1=0$$

If

$$\left.\begin{aligned} F=G=H&=\tfrac{1}{2\sigma_{yo}^2}\\ L=M=N&=\tfrac{3}{2\sigma_{yo}^2}\end{aligned}\right\}\quad \text{von Mises}$$

In general

$$F=\tfrac{1}{2}\left[\tfrac{1}{(\sigma_{yo}^{11})^2}+\tfrac{1}{(\sigma_{yo}^{22})^2}-\tfrac{1}{(\sigma_{yo}^{33})^2}\right]$$
$$G=\tfrac{1}{2}\left[\tfrac{1}{(\sigma_{yo}^{11})^2}-\tfrac{1}{(\sigma_{yo}^{22})^2}+\tfrac{1}{(\sigma_{yo}^{33})^2}\right]$$
$$H=\tfrac{1}{2}\left[-\tfrac{1}{(\sigma_{yo}^{11})^2}+\tfrac{1}{(\sigma_{yo}^{22})^2}+\tfrac{1}{(\sigma_{yo}^{33})^2}\right]$$
$$L=\tfrac{1}{2(\sigma_{yo}^{12})^2}\quad M=\tfrac{1}{2(\sigma_{yo}^{13})^2}\quad N=\tfrac{1}{2(\sigma_{yo}^{23})^2}$$

Restrictions on degree of orthotropy

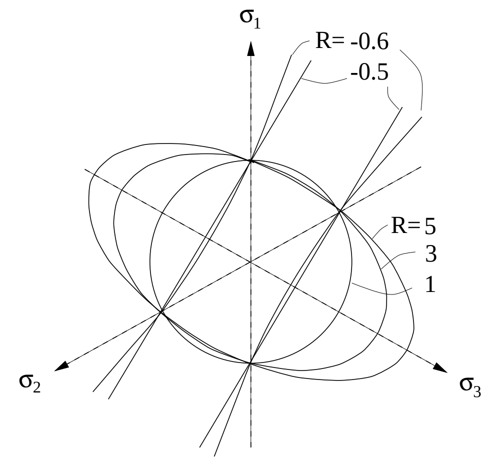
$$\frac{4}{(\sigma_{yo}^{11})^2(\sigma_{yo}^{22})^2}>\left[\frac{1}{(\sigma_{yo}^{33})^2}-\frac{1}{(\sigma_{yo}^{11})^2}-\frac{1}{(\sigma_{yo}^{22})^2}\right]^2$$

The reason is? We have assumed a quadratic form



## HILL'S CRITERION

Let material axes of orthotropy coincide with the principal axes




where

$$\sigma_{yo}^{11}=\sigma_{yo}^{22}\qquad (\sigma_{yo}^{33})^2=\frac{1}{2}(\sigma_{yo}^{11})^2(1+R)$$

For $R=1$ $\Rightarrow$ von Mises

Problems when $R\to-\frac{1}{2}$

## ANISOTROPIC CRITERIA
## – CALIBRATION PAPERBOARD –

From Persson (1991) we have

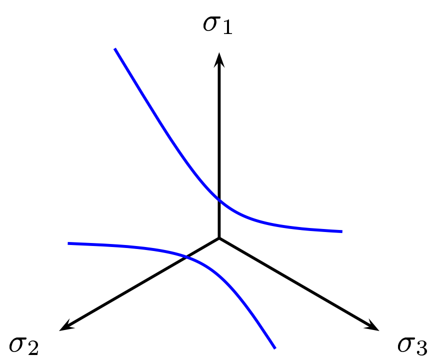
$$\sigma_{yo}^{11} = 12\text{MPa} \qquad \sigma_{yo}^{22} = 6\text{MPa} \qquad \sigma_{yo}^{33} = 0.11\text{MPa}$$

Parameters in Hill's criterion

$$F = \frac{1}{2}\left[\frac{1}{(\sigma_{yo}^{11})^2} + \frac{1}{(\sigma_{yo}^{22})^2} - \frac{1}{(\sigma_{yo}^{33})^2}\right] = -49.9826$$

$$G = \frac{1}{2}\left[\frac{1}{(\sigma_{yo}^{11})^2} - \frac{1}{(\sigma_{yo}^{22})^2} + \frac{1}{(\sigma_{yo}^{33})^2}\right] = 49.9896$$

$$H = \frac{1}{2}\left[-\frac{1}{(\sigma_{yo}^{11})^2} + \frac{1}{(\sigma_{yo}^{22})^2} + \frac{1}{(\sigma_{yo}^{33})^2}\right] = 50.014$$




Restrictions on degree of orthotropy

$$\frac{4}{(\sigma_{yo}^{11})^2(\sigma_{yo}^{22})^2} > \left[\frac{1}{(\sigma_{yo}^{33})^2} - \frac{1}{(\sigma_{yo}^{11})^2} - \frac{1}{(\sigma_{yo}^{22})^2}\right]^2$$



## ANISOTROPIC YIELD CRITERIA

We have in general

$$f(\sigma_{ij}) = 0$$

Example: Assuming orthotropy

$$\sigma_{ij}P_{ijkl}\sigma_{kl} - 1 = 0 \quad \text{or} \quad \boldsymbol{\sigma}^T\boldsymbol{P}\boldsymbol{\sigma} - 1 = 0$$

where

$$\boldsymbol{P} = \begin{bmatrix} A & -F & -G & 0 & 0 & 0 \\ -F & B & -H & 0 & 0 & 0 \\ -G & -H & C & 0 & 0 & 0 \\ 0 & 0 & 0 & 2L & 0 & 0 \\ 0 & 0 & 0 & 0 & 2M & 0 \\ 0 & 0 & 0 & 0 & 0 & 2N \end{bmatrix}$$

Assuming orthotropy and pressure independent, Hill (1948,1950)

$$s_{ij}P_{ijkl}s_{kl} - 1 = 0 \quad \text{or} \quad \boldsymbol{s}^T\boldsymbol{P}\boldsymbol{s} - 1 = 0$$

Generalize

$$\sigma_{ij}P_{ijkl}\sigma_{kl} + \sigma_{ij}q_{ij} - 1 = 0$$
or
$$\boldsymbol{\sigma}^T\boldsymbol{P}\boldsymbol{\sigma} + \boldsymbol{\sigma}^T\boldsymbol{q} - 1 = 0$$

Tsai-Wu anisotropic yield criterion (1971)