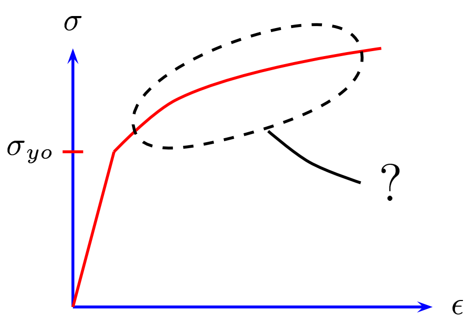
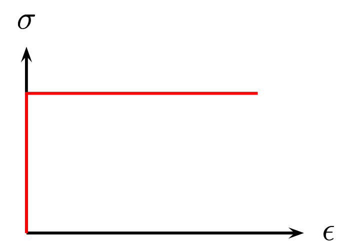
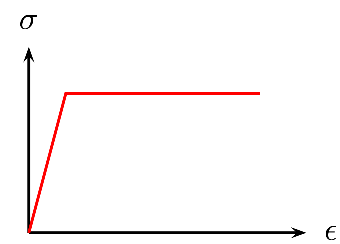
INTRODUCTORY REMARKS TO THE PLASTICITY THEORY



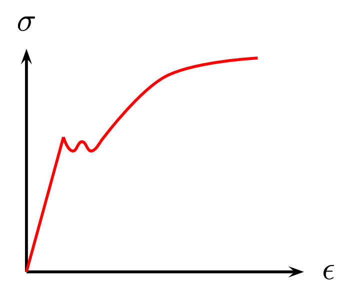
SIMPLIFIED MODELS



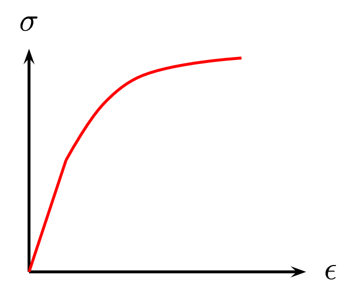
stiff-ideal plast.



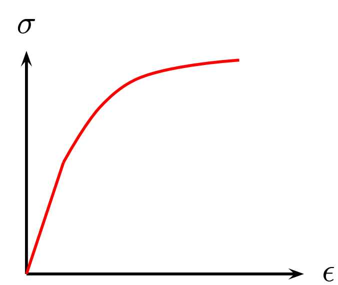
elastic-ideal plast.



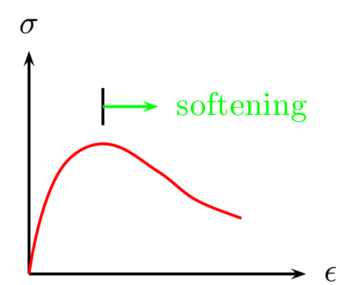
mild steel



alloyed steel
-aluminum



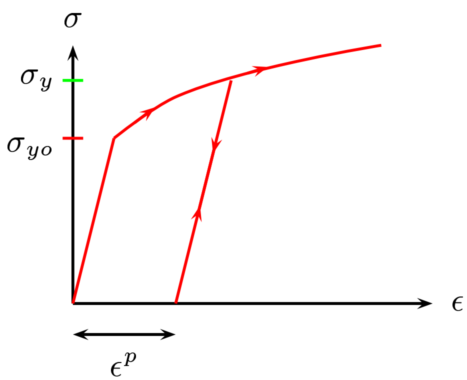
hardening plast.



concrete
-rocks

Div. of Solid Mechanics, Lund University

DEFINITIONS



$\sigma_{yo} =$ initial yield stress

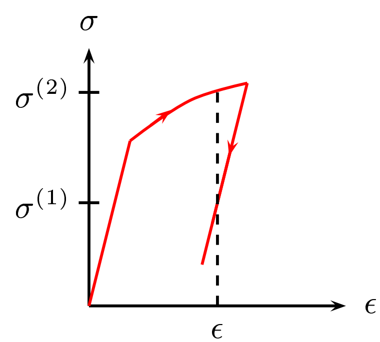
$\sigma_y =$ current yield stress

$\sigma_{yo} \longrightarrow$ initial yield function

$\sigma_y \longrightarrow$ current yield function

The manner in which the current yield function evolves with plastic deformation is called the hardening rule

HISTORY DEPENDENCE



same ϵ

– two different σ

no unique relation between σ and ϵ

MISSING INFORMATION

DEFINITIONS CONT.

Initial yield surface

$$F(\sigma_{ij}) = 0$$

Current yield surface

$$f(\sigma_{ij}, \underbrace{K^\alpha}_{\text{hardening parameters}}) = 0$$

$$K^\alpha = \{K^1, K^2, \cdots\}$$

$$K^\alpha = 0 \quad \text{initially}$$

$$f(\sigma_{ij}, 0) = F(\sigma_{ij})$$

Hardening rule =
rule for how the yield surface changes with the plastic loading

Choice of hardening parameters =
choice of hardeing rule

DEFINITIONS CONT.

Internal variables

$$\kappa^\alpha = \{\kappa^1, \kappa^2, \cdots\}$$

κ^α characterizes the state of the elasto-plastic material

Internal variables = state variables

$$\kappa^\alpha = 0 \quad \text{initially}$$

Example: choose ϵ^p_{ij} as internal variables

$$K^\alpha = K^\alpha(\kappa^\beta)$$

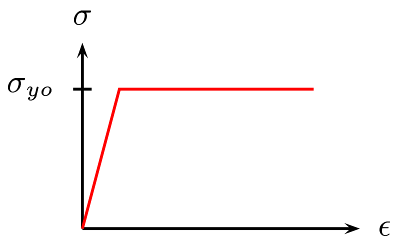
i.e.

$$\dot{K}^\alpha = \frac{\partial K^\alpha}{\partial \kappa^\beta} \dot{\kappa}^\beta$$

$$\dot{\kappa}^\beta = 0 \quad \text{for elastic behaviour}$$

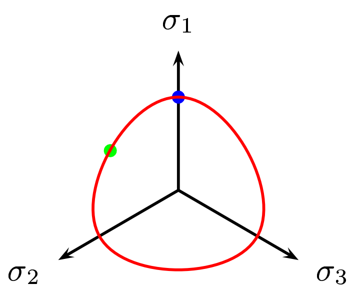
$$\Rightarrow \quad \dot{K}^\alpha = 0 \quad \text{for elastic behaviour}$$

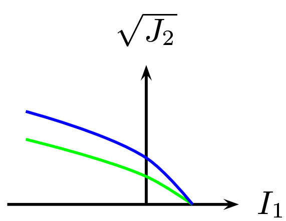
IDEAL PLASTICITY



Yield stress unaffected by plasticity

Generalization





Current yield surface fixed in stress space

$$F(I_1, J_2, cos3\theta) = 0 \qquad \text{(isotropic)}$$

or

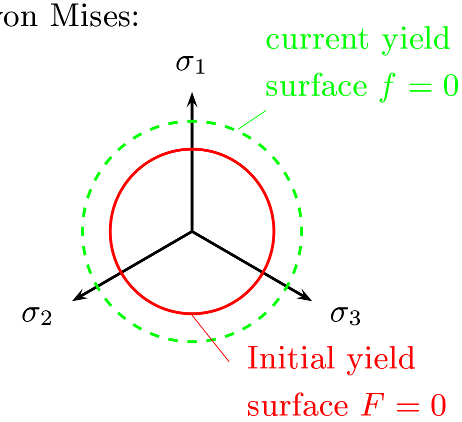
$$f(\sigma_{ij}, K^\alpha) = F(\sigma_{ij}) = 0$$

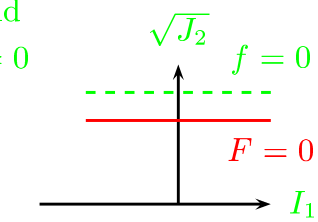
no dependence on hardening parameter

ISOTROPIC HARDENING

Shape an position remain fixed – but size of yield surface changes with the loading

von Mises:





Initial yield surface:

$$F(\sigma_{ij}) = \sqrt{3J_2} - \sigma_{yo} = 0$$

Current yield surface:

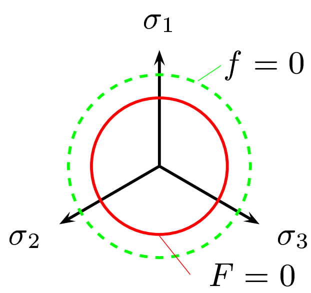
$$f(\sigma_{ij}, K) = \sqrt{3J_2} - \sigma_{yo} - K = 0$$

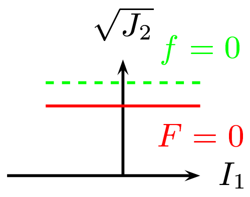
i.e.

$$\underbrace{f(\sigma_{ij}, K)}_{current} = \underbrace{F(\sigma_{ij})}_{initial} - K = 0$$

ISOTROPIC HARDENING

von Mises:





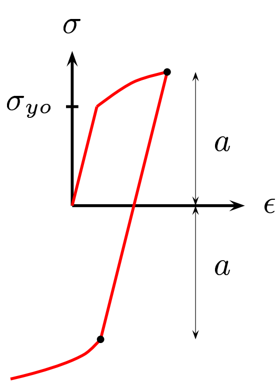
Current yield surface can be written

$$f(\sigma_{ij}, K^{\alpha}) = F(\sigma_{ij}) - K = \sqrt{3J_2} - \sigma_{yo} - K = 0$$

K determines the expansion of the yield surface

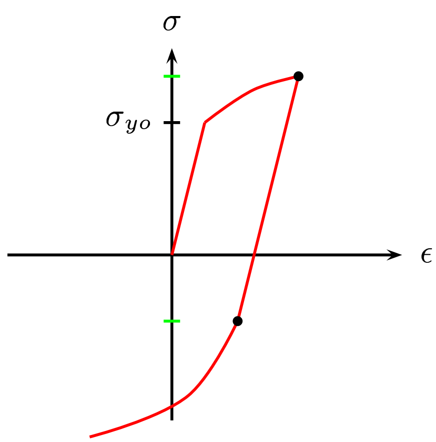
i.e. uniform (=isotropic) expansion of yield surface

Effects of reversed loading



i.e. current yield stress in tension
=
current yield stress in compression

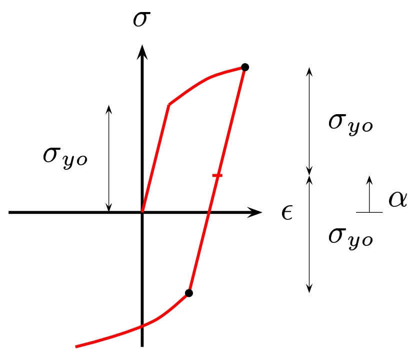
IN REALITY – FOR STEEL



current yield stress in tension $\neq$ current yield stress in compression

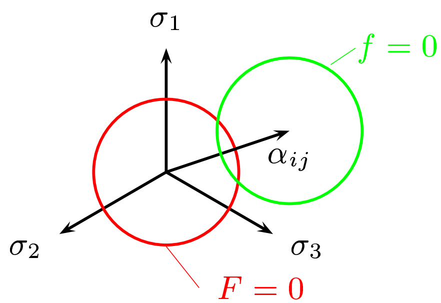
Bauchinger-effect

KINEMATIC HARDENING



Size of current yield surface is constant
= kinematic hardening

von Mises



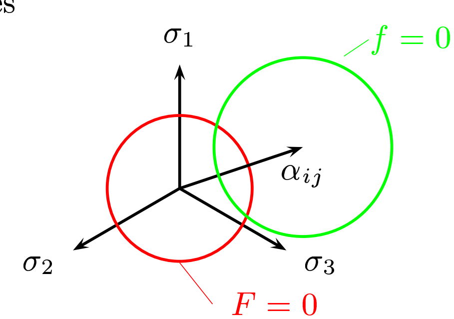
current yield surface can be written as

$$f(\sigma_{ij}, K) = F(\sigma_{ij} - \alpha_{ij}) = 0, \quad K = \{\alpha_{ij}\}$$

"back-stress" – depends on plastic history

MIXED HARDENING

von Mises



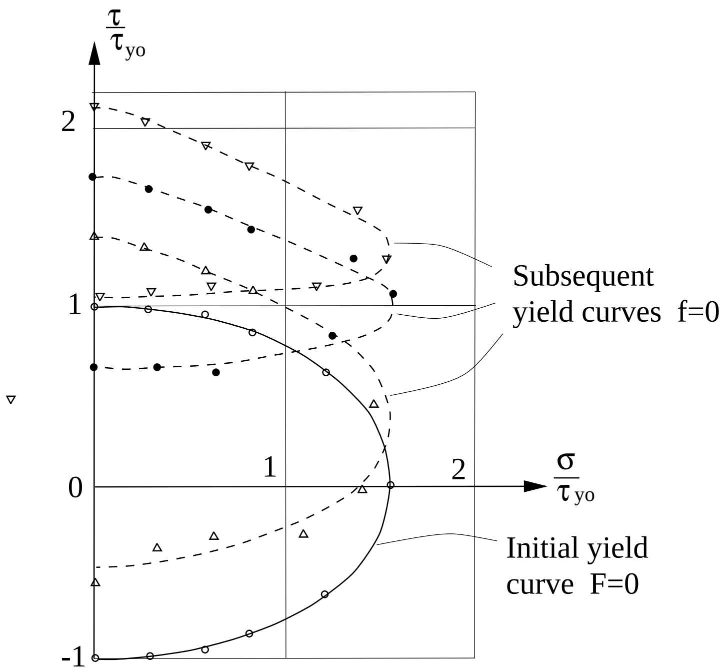
current yield surface moves (kinematic) and expands (isotropic) =mixed hardening

current yield surface can be written as

$$f(\sigma_{ij}, K^{\alpha}) = \underbrace{F(\sigma_{ij} - \alpha_{ij})}_{kinematic} - \underbrace{K}_{isotropic} = 0$$

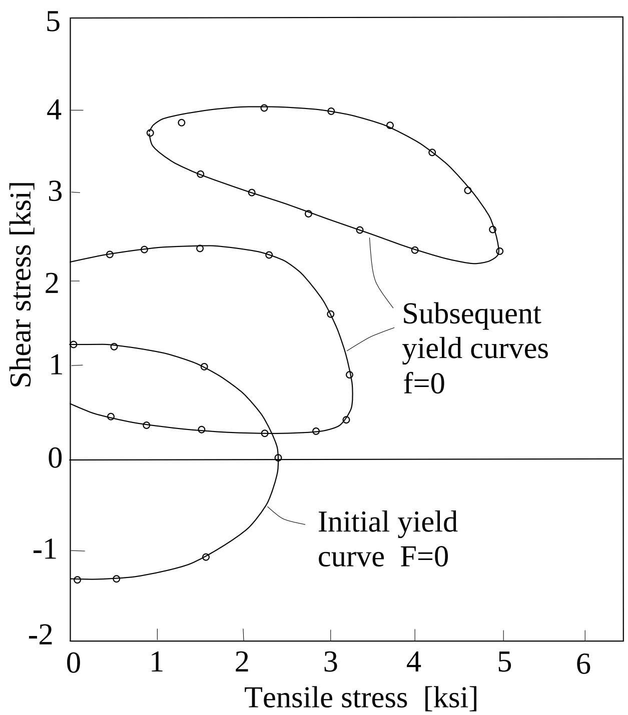
i.e. $K^{\alpha} = \{\alpha_{ij}, K\}$

DISTORTIONAL HARDENING – ANISOTROPIC HARDENING



combined torsion and tension tests Ivey (1961)

DISTORTIONAL HARDENING – ANISOTROPIC HARDENING



combined torsion and tension tests Phillips and Tang (1972)

CONSTITUTIVE RELATIONS

In general

$$\epsilon_{ij} = \epsilon_{ij}^e + \epsilon_{ij}^p$$

Hookes laws

$$\sigma_{ij} = D_{ijkl}\epsilon_{kl}^e \quad \text{or} \quad \epsilon_{ij}^e = C_{ijkl}\sigma_{kl}$$

We need an expression for ϵ_{ij}^p in the form

$$d\epsilon_{ij}^p = ?$$

or

$$\dot{\epsilon}_{ij}^p = ?$$

Metals and Steel

Plastic strains:

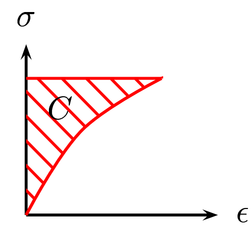
- no influence of I_1

only deviatoric stresses influence plasticity

- volumetric behaviour $\epsilon_{ii} = \epsilon_{ii}^e + \epsilon_{ii}^p$ is elastic

$$\epsilon_{ii}^p = 0$$

HYPER-ELASTICITY

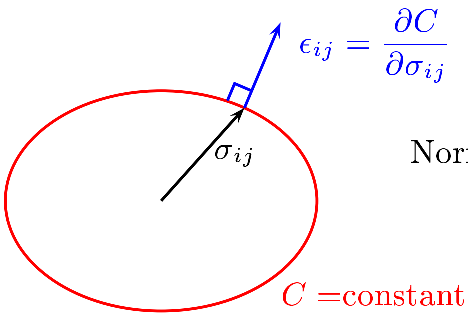


C=complementary strain energy

$C = C(\sigma_{ij})$

strain tensor can be derived from the potential C

$$\epsilon_{ij} = \frac{\partial C}{\partial \sigma_{ij}}$$



Normality principle

$C(\sigma_{ij})$ is convex if $\dfrac{\partial^2 C}{\partial \sigma_{ij} \partial \sigma_{kl}}$ is positive definite

C is convex

EVOLUTION OF PLASTIC STRAINS

Good arguments for

$$\dot{\epsilon}_{ij}^p = \dot{\lambda}\frac{\partial f}{\partial \sigma_{ij}}, \quad \dot{\lambda} \geq 0 \qquad \text{assoc. plasticity}$$

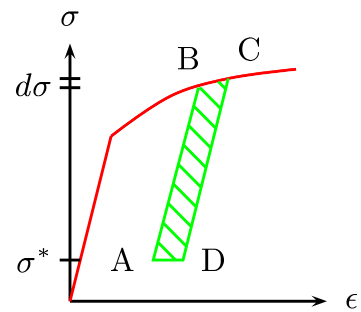
but no fundamental principle

$$\dot{\epsilon}_{ij}^p = \dot{\lambda}\frac{\partial g}{\partial \sigma_{ij}}, \quad \dot{\lambda} \geq 0 \qquad \text{non-assoc. plasticity}$$

is acceptable

(more possibilities?)

DRUCKER'S POSTULATE (1951)



Postulate

$$\int_{ABCD} (\sigma_{ij} - \sigma_{ij}^*) d\epsilon_{ij} \geq 0$$

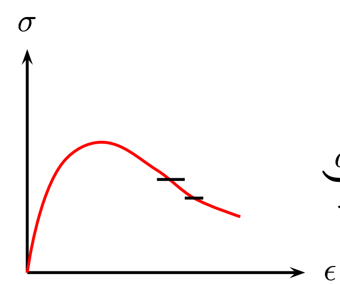
during a stress cycle

using $\epsilon = \epsilon_{ij}^e + \epsilon_{ij}^p$ and trapezoidal rule

$$(\sigma_{ij} - \sigma_{ij}^*) d\epsilon_{ij}^p + \frac{1}{2} d\sigma_{ij} d\epsilon_{ij}^p \geq 0$$

Choose $\sigma_{ij} = \sigma_{ij}^* \quad \Rightarrow \quad d\sigma_{ij} d\epsilon_{ij}^p \geq 0$

i.e. it also follows that $(\sigma_{ij} - \sigma_{ij}^*) d\epsilon_{ij}^p \geq 0$



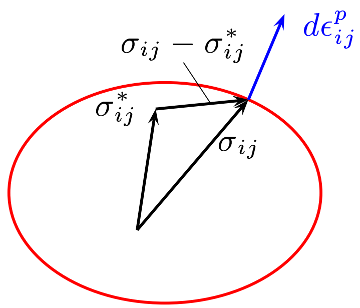
$$\underbrace{d\sigma}_{<0}\underbrace{d\epsilon^p}_{>0} \geq 0$$ Contradiction

Postulate holds for hardening materials

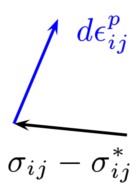
DRUCKER'S POSTULATE (1951)

We found

$$(\sigma_{ij} - \sigma^*_{ij}) d\epsilon^P_{ij} \geq 0$$



σ^*_{ij} arbitrary inside or on the yield surface



cannot occur $\Rightarrow$ Convexity of yield surface

If yield surface is smooth $\dot{\epsilon}^p_{ij} \sim \frac{\partial f}{\partial \sigma_{ij}}$

normality $\Rightarrow$ $\dot{\epsilon}^p_{ij} = \dot{\lambda} \frac{\partial f}{\partial \sigma_{ij}}$

EVOLUTION OF PLASTIC STRAINS

We found

Drucker's postulate

$$(\sigma_{ij} - \sigma^*_{ij}) d\epsilon^p_{ij} \geq 0$$

convexity and normality

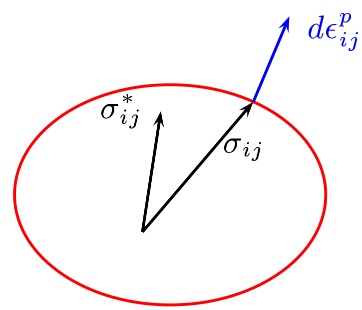
Derived by Drucker assuming hardening and $\int_{ABCD} (\sigma_{ij} - \sigma^*_{ij}) d\epsilon_{ij} \geq 0$

But it holds even for ideal and softening plasticity

Postulate $(\sigma_{ij} - \sigma^*_{ij}) d\epsilon^p_{ij} \geq 0$

von Mises (1928), Taylor (1947), Hill (1948)

implies convexity and normality for hardening, ideal and softening plasticity



Define $D = \sigma_{ij} d\epsilon^p_{ij}$

$D^* = \sigma^*_{ij} d\epsilon^p_{ij}$

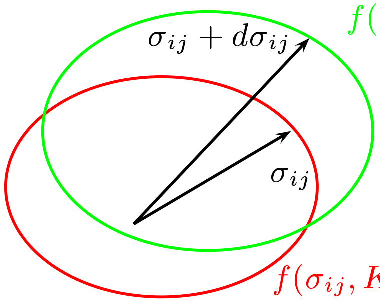
$D \geq D^*$

postulate of max. plastic dissipation

CONSISTENCY RELATION

We found

$$\dot{\epsilon}^p_{ij} = \underbrace{\dot{\lambda}}_{?} \frac{\partial g}{\partial \sigma_{ij}}$$



$f = 0$ during plastic loading

USE OF CONSISTENCY RELATION

We found

yield criterion
$f(\sigma_{ij}, K^\alpha) = 0$

flow rule
$\dot{\epsilon}^p_{ij} = \dot{\lambda} \frac{\partial g}{\partial \sigma_{ij}}$

Consistency relation $\dot{f} = 0$

$$\frac{\partial f}{\partial \sigma_{ij}} \dot{\sigma}_{ij} + \frac{\partial f}{\partial K^\alpha} \dot{K}^\alpha = 0$$

But $K^\alpha = K^\alpha(\kappa^\beta)$

$$\frac{\partial f}{\partial \sigma_{ij}} \dot{\sigma}_{ij} + \frac{\partial f}{\partial K^\alpha} \frac{\partial K^\alpha}{\partial \kappa^\beta} \dot{\kappa}^\beta = 0$$

Evolution laws, $\dot{\kappa}^\beta$ must depend on $\dot{\epsilon}^p_{ij}$

$$\dot{\kappa}^\beta = a^\beta(\dot{\epsilon}^p_{ij}, K^\alpha) = a^\beta(\dot{\lambda}, \frac{\partial g}{\partial \sigma_{ij}}, K^\alpha)$$

homogenous in time

$$\dot{\kappa}^\beta = \dot{\lambda} \underbrace{k^\beta(\sigma_{ij}, K^\alpha)}$$

evolution function
(that we choose)

$$\frac{\partial f}{\partial \sigma_{ij}} \dot{\sigma}_{ij} + \underbrace{\frac{\partial f}{\partial K^\alpha} \frac{\partial K^\alpha}{\partial \kappa^\beta} k^\beta}_{-H} \dot{\lambda} = 0$$

$-H$ = generalized plastic modulus

STRESS DRIVEN FORMAT

We found

$$\frac{\partial f}{\partial \sigma_{ij}}\dot{\sigma}_{ij} - H\dot{\lambda} = 0, \qquad H = -\frac{\partial f}{\partial K^\alpha}\frac{\partial K^\alpha}{\partial \kappa^\beta}k^\beta$$

i.e.

$$\dot{\lambda} = \frac{1}{H}\frac{\partial f}{\partial \sigma_{kl}}\dot{\sigma}_{kl} \qquad (H \neq 0)$$

Evolution of plastic strains

$$\dot{\epsilon}^p_{ij} = \frac{1}{H}\frac{\partial g}{\partial \sigma_{ij}}\frac{\partial f}{\partial \sigma_{kl}}\dot{\sigma}_{kl}$$

Hooke's law

$$\dot{\epsilon}^e_{ij} = C_{ijkl}\dot{\sigma}_{kl}$$

Total strain rate

$$\dot{\epsilon}_{ij} = \dot{\epsilon}^e_{ij} + \dot{\epsilon}^p_{ij}$$

or

$$\dot{\epsilon}_{ij} = (C_{ijkl} + \frac{1}{H}\frac{\partial g}{\partial \sigma_{ij}}\frac{\partial f}{\partial \sigma_{kl}})\dot{\sigma}_{kl}$$

CONSTITUTIVE RELATIONS FOR ELASTO-PLASTICITY

Total strain

$$\epsilon_{ij} = \epsilon^e_{ij} + \epsilon^p_{ij}$$

Hooke's law

$$\epsilon^e_{ij} = C_{ijkl}\sigma_{kl}$$

Flow rule

$$\dot{\epsilon}^p_{ij} = \dot{\lambda}\frac{\partial g}{\partial \sigma_{ij}} \qquad \dot{\lambda} \geq 0$$

where $g = g(\sigma_{ij}, K^\alpha)$

Yield function

$$f = f(\sigma_{ij}, K^\alpha)$$

during plastic loading $f = 0$.

Hardening laws

$$K^\alpha = K^\alpha(\kappa^\beta)$$

Evolution laws

$$\dot{\kappa}^\beta = \dot{\lambda}k^\beta$$

where $k^\beta = k^\beta(\sigma_{ij}, K^\alpha)$

VON MISES ISOTROPIC HARDENING

Yield criteria

$$f(\sigma_{ij}, K) = \sqrt{3J_2} - \sigma_{yo} - K = 0$$

define $\sigma_y(\kappa) = \sigma_{yo} + K(\kappa)$

$$f = \sqrt{3J_2} - \sigma_y(\kappa) = 0$$

Flow rule (ass. plast. $\Rightarrow f = g$)

$$\dot{\epsilon}^p_{ij} = \dot{\lambda}\frac{\partial f}{\partial \sigma_{ij}} = \dot{\lambda}\frac{3s_{ij}}{2\sigma_y}$$

Define rate of effective plastic strain

$$\dot{\epsilon}^p_{eff} = \left(\frac{2}{3}\dot{\epsilon}^p_{ij}\dot{\epsilon}^p_{ij}\right)^{1/2} \quad \Rightarrow \quad \dot{\epsilon}^p_{eff} = \dot{\lambda}$$

Define effective stress

$$\sigma_{eff} = \sqrt{3J_2} = \sqrt{\frac{3}{2}s_{ij}s_{ij}}$$

It then follows that the rate of plastic work can be written as

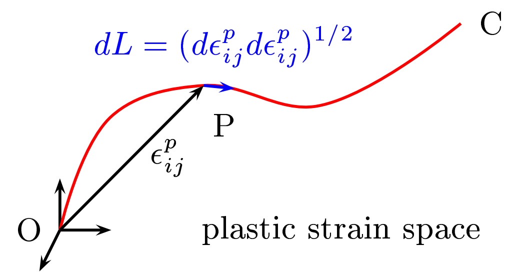
$$\dot{W}^p = \sigma_{ij}\dot{\epsilon}^p_{ij} = \sigma_{eff}\dot{\epsilon}^p_{eff}$$

and the yield criteria as

$$f = \sigma_{eff} - \sigma_y(\kappa) = 0$$

VON MISES ISOTROPIC HARDENING, CONT.

Choice of internal variable κ



Arc length

$$\text{OPC} = \int_O^C dL = \int_O^C (d\epsilon^p_{ij} d\epsilon^p_{ij})^{1/2} = \sqrt{\frac{3}{2}}\int_O^C d\epsilon^p_{eff}$$

Define

$$\epsilon^p_{eff} = \int_O^C d\epsilon^p_{eff}$$

expresses the plastic strain history

Choose the evolution law

$$\dot{\kappa} = \dot{\epsilon}^p_{eff} \quad \Rightarrow \quad k = 1$$

strain hardening Odquist (1938)

VON MISES ISOTROPIC HARDENING, CONT.

Identifying the hardening law, uniaxial tension

$$[\sigma_{ij}] = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Deviatoric stresses $I_1 = \sigma$

$$[s_{ij}] = [\sigma_{ij}] - \frac{1}{3} I_1 [\delta_{ij}] = \begin{bmatrix} \frac{2}{3}\sigma & 0 & 0 \\ 0 & -\frac{1}{3}\sigma & 0 \\ 0 & 0 & -\frac{1}{3}\sigma \end{bmatrix}$$

Effective stress

$$\sigma_{eff} = \sqrt{\frac{3}{2}(s_1^2 + s_2^2 + s_3^2)} = \sigma$$

Flow rule

$$[\dot{\epsilon}^p_{ij}] = \dot{\lambda} \frac{3}{2\sigma_y} \begin{bmatrix} \frac{2}{3}\sigma & 0 & 0 \\ 0 & -\frac{1}{3}\sigma & 0 \\ 0 & 0 & -\frac{1}{3}\sigma \end{bmatrix} = \dot{\lambda} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix}$$

Rate of effective plastic strain

$$\dot{\epsilon}^p_{eff} = \sqrt{\frac{2}{3}((\dot{\epsilon}^p_{11})^2 + (\dot{\epsilon}^p_{22})^2 + (\dot{\epsilon}^p_{33})^2)} = \dot{\lambda} = \dot{\epsilon}^p_{11} = \dot{\epsilon}^p = \dot{\kappa}$$

For uniaxial loading $\kappa = \epsilon^p$

$$K = K(\epsilon^p)$$

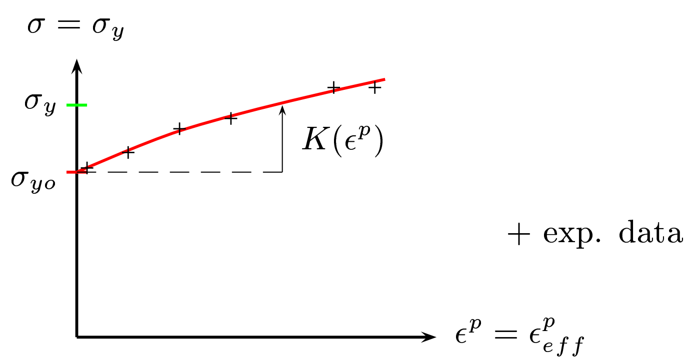
VON MISES ISOTROPIC HARDENING, CONT.

Identifying the hardening law, uniaxial tension

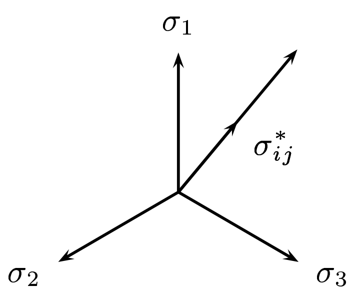
$$\sigma_{eff} = \sigma \qquad \epsilon^p_{eff} = \epsilon^p$$

Yield function

$$f = \sigma - \underbrace{\sigma_{yo} - K(\epsilon^p)}_{\sigma_y} = 0$$



VON MISES PROPORTIONAL LOADING



$$\sigma_{ij} = \beta(t)\sigma_{ij}^*$$
$$s_{ij} = \beta(t)s_{ij}^*$$
$$\sigma_{kk} = \beta(t)\sigma_{kk}^*$$
$$\sigma_{eff} = \beta(t)\sigma_{eff}^*$$

During plastic loading

$$\dot{\epsilon}_{ij}^p = \dot{\lambda}\frac{3s_{ij}}{2\sigma_{eff}} = \dot{\lambda}\frac{3s_{ij}^*}{2\sigma_{eff}^*}$$

Integration

$$\epsilon_{ij}^p = \lambda\frac{3s_{ij}^*}{2\sigma_{eff}^*} = \lambda\frac{3s_{ij}}{2\sigma_{eff}} = e_{ij}^p$$

Effective plastic strain

$$\epsilon_{eff}^p = \lambda$$

Isotropic elasticity

$$\epsilon_{kk}^e = \frac{1}{3K}\sigma_{kk} \qquad e_{ij}^e = \frac{1}{2G^e}s_{ij}$$

VON MISES PROPORTIONAL LOADING

Total strain

$$\epsilon_{kk} = \epsilon_{kk}^e + \epsilon_{kk}^p = \frac{\sigma_{kk}}{3K}$$

$$e_{ij} = e_{ij}^e + e_{ij}^p = \left(\frac{1}{2G^e} + \frac{3\epsilon_{eff}^p}{2\sigma_{eff}}\right)s_{ij}$$

or

$$\sigma_{kk} = 3K\epsilon_{kk}$$

$$s_{ij} = 2Ge_{ij}$$

where

$$K = constant$$
$$G(J_2) = \frac{1}{2}\frac{1}{\frac{1}{2G^e} + \frac{3\epsilon_{eff}^p}{2\sigma_{eff}}}$$

where we used that $\sigma_{eff} = \sigma_{eff}(\epsilon_{eff}^p) = \sqrt{3J_2}$

Deformation plasticity or Nonlinear isotropic Hooke formulation

Hyper-elasticity?

$$\frac{\partial}{\partial J_2}(\frac{\partial C}{\partial I_1}) = \frac{\partial}{\partial I_1}(\frac{\partial C}{\partial J_2})$$

$$\frac{\sigma_{kk}}{3}\frac{\partial}{\partial J_2}(\frac{1}{3K}) = \frac{\partial}{\partial I_1}(\frac{1}{2G})$$