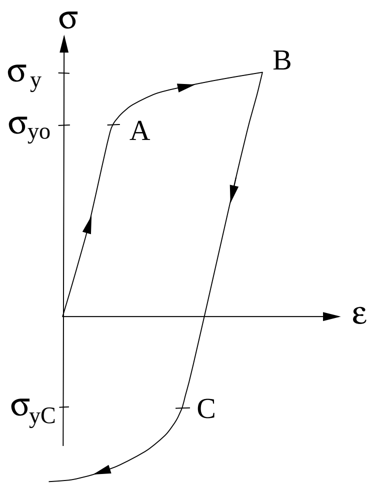
GENERAL EXPERIMENTAL EVIDENCE

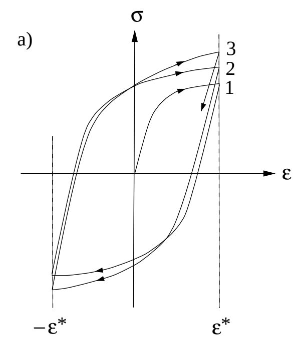
Bauschinger-effect



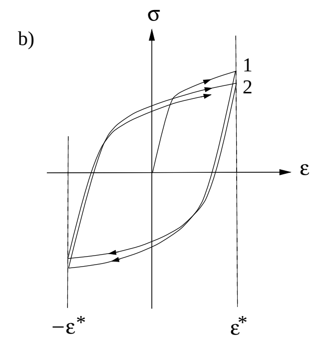
GENERAL EXPERIMENTAL EVIDENCE

Strain cyclings

cyclic hardening

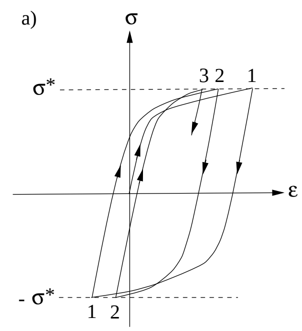


cyclic softening

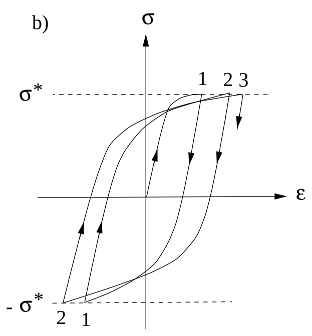


Stress cycling

cyclic hardening

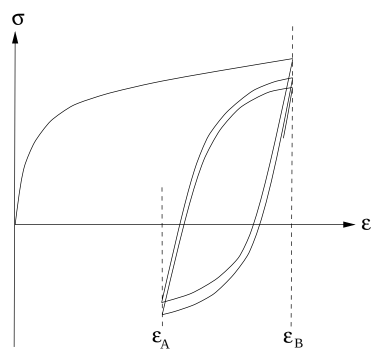


cyclic softening

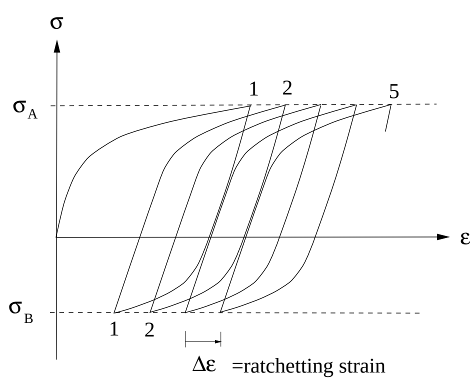


GENERAL EXPERIMENTAL EVIDENCE

Strain cyclings between unsymmetric strain values

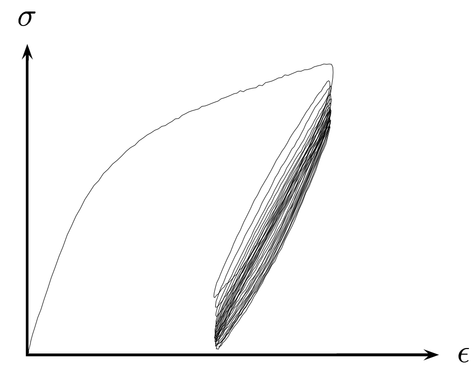


Stress cycling between unsymmetric stress values

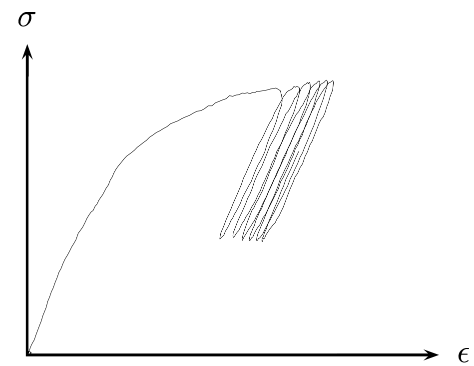


GENERAL EXPERIMENTAL EVIDENCE, PAPERBOARD

Strain cyclings between unsymmetric strain values

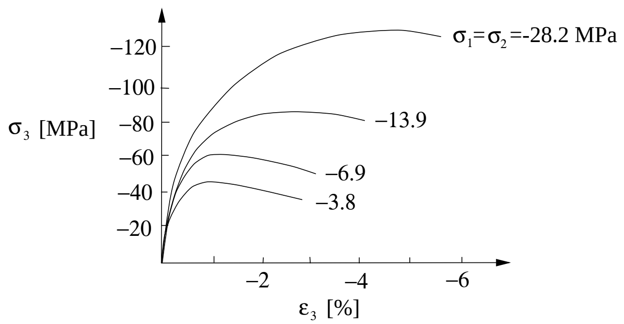


Stress cycling between unsymmetric stress values

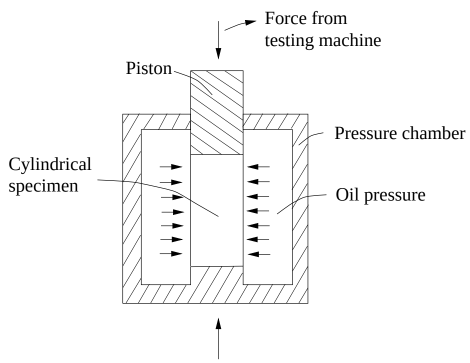


GENERAL EXPERIMENTAL EVIDENCE

Triaxial compression of concrete

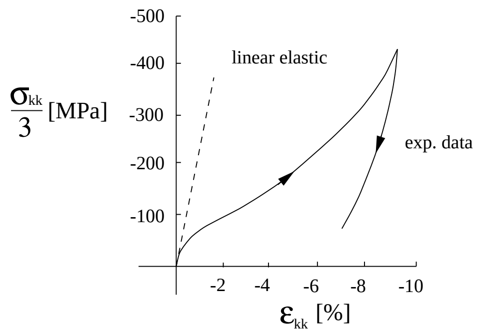


von Kármán pressure cell

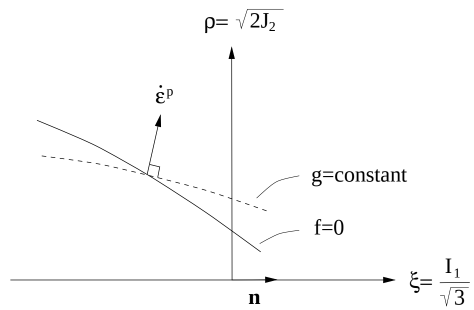


GENERAL EXPERIMENTAL EVIDENCE

Hydrostatic compression of concrete $\sigma_1 = \sigma_2 = \sigma_3 < 0$



Meridian plane, plastic volume increase



$$\dot{\epsilon}^p_{ii} = 3\dot{\lambda}\frac{\partial g}{\partial I_1}$$

ISOTROPIC HARDENING OF VON MISES MATERIAL

We have

$$\dot{\sigma}_{ij} = D^{ep}_{ijkl}\dot{\epsilon}_{kl}$$

where

$$D^{ep}_{ijkl} = D_{ijkl} - \frac{1}{A} D_{ijst}\frac{\partial g}{\partial \sigma_{st}}\frac{\partial f}{\partial \sigma_{mn}} D_{mnkl}$$

and

$$A = H + \frac{\partial g}{\partial \sigma_{ij}} D_{ijkl}\frac{\partial f}{\partial \sigma_{kl}}$$

von Mises – isotropic hardening

$$g = f = \sqrt{\frac{3}{2}s_{ij}s_{ij}} - \sigma_{yo} = 0$$

where

$$\sigma_y(\kappa) = \sigma_{yo} + K(\kappa)$$

Isotropic elasticity

$$D_{ijkl} = 2G[\frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \frac{\nu}{1-2\nu}\delta_{ij}\delta_{kl}]$$

We find

$$D^{ep}_{ijkl} = D_{ijkl} - \frac{9G^2}{A}\frac{s_{ij}s_{kl}}{\sigma_y^2}$$

ISOTROPIC HARDENING OF VON MISES MATERIAL

We have

$$D^{ep}_{ijkl} = D_{ijkl} - \frac{9G^2}{A}\frac{s_{ij}s_{kl}}{\sigma_y^2}$$

or in matrix format

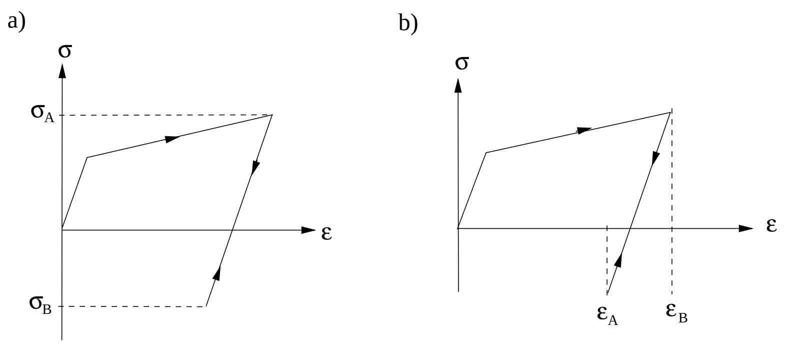
$$\boldsymbol{D}^{ep} = \frac{E}{1+\nu}\begin{bmatrix} \frac{1-\nu}{1-2\nu} - Ms_{11}^2 & \frac{\nu}{1-2\nu} - Ms_{11}s_{22} & \frac{\nu}{1-2\nu} - Ms_{11}s_{33} & -Ms_{11}s_{12} & -Ms_{11}s_{13} & -Ms_{11}s_{23} \\ & \frac{1-\nu}{1-2\nu} - Ms_{22}^2 & \frac{\nu}{1-2\nu} - Ms_{22}s_{33} & -Ms_{22}s_{12} & -Ms_{22}s_{13} & -Ms_{22}s_{23} \\ & & \frac{1-\nu}{1-2\nu} - Ms_{33}^2 & -Ms_{33}s_{12} & -Ms_{33}s_{13} & -Ms_{33}s_{23} \\ & & & \frac{1}{2} - Ms_{12}^2 & -Ms_{12}s_{13} & -Ms_{12}s_{23} \\ & & & & \frac{1}{2} - Ms_{13}^2 & -Ms_{13}s_{23} \\ & & & & & \frac{1}{2} - Ms_{23}^2 \end{bmatrix}$$

$$M = \frac{9G}{2A\sigma_y^2} \qquad A = H + 3G$$

ISOTROPIC HARDENING OF VON MISES MATERIAL

Unsymmetric loading

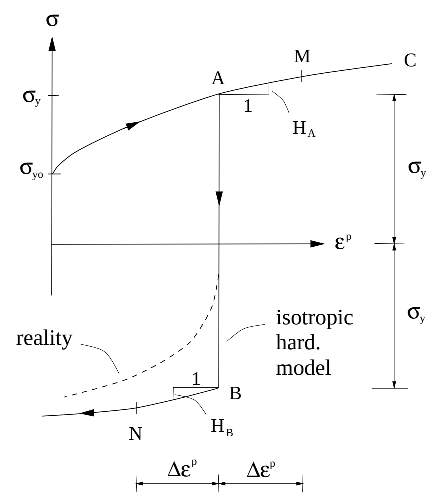
unsymmetric stress cycling   unsymmetric stress cycling



no ratchetting   no mean stess relaxation

ISOTROPIC HARDENING OF VON MISES MATERIAL

Nonlinear isotropic hardening



$$H = \frac{d\sigma_y(\epsilon^p_{eff})}{d\epsilon^p_{eff}}$$

KINEMATIC HARDENING OF VON MISES MATERIAL

$$f(\sigma_{ij}, K^{\alpha}) = F(\bar{J}_2) = 0$$

where

$$\bar{J}_2 = \frac{1}{2}\bar{s}_{ij}\bar{s}_{ij} = \frac{1}{2}(s_{ij} - \alpha^d_{ij})(s_{ij} - \alpha^d_{ij})$$

or

$$\boxed{f = \sqrt{\tfrac{3}{2}(s_{ij} - \alpha^d_{ij})(s_{ij} - \alpha^d_{ij})} - \sigma_{yo} = 0}$$

Assume Melan (1938)-Prager (1955) evolution law for back-stress

$$\boxed{\dot{\alpha}_{ij} = c\dot{\epsilon}^p_{ij}}$$

Flow rule

$$\dot{\epsilon}^p_{ij} = \dot{\lambda}\frac{\partial f}{\partial \sigma_{ij}} = \dot{\lambda}\frac{3(s_{ij} - \alpha^d_{ij})}{2\sigma_{yo}}$$

i.e.

$$\dot{\alpha}_{ij} = \dot{\alpha}^d_{ij}$$

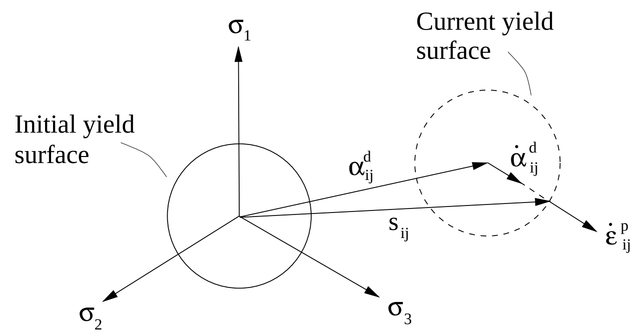
purely deviatoric

Generalized plastic modulus

$$H = \frac{3}{2}c$$

KINEMATIC HARDENING OF VON MISES MATERIAL

Illustraition of Melan-Prager's evolution law



KINEMATIC HARDENING OF VON MISES MATERIAL

We have

$$D^{ep}_{ijkl} = D_{ijkl} - \frac{9G^2}{A}\frac{\bar{s}_{ij}\bar{s}_{kl}}{\sigma_y^2}$$

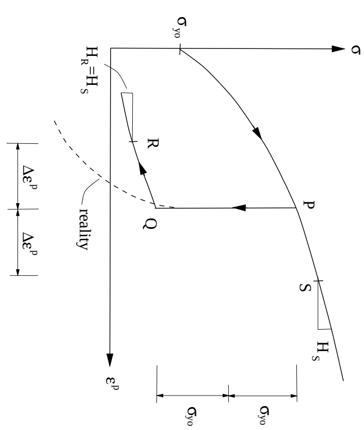
or in matrix format

$$\boldsymbol{D}^{ep} = \frac{E}{1+\nu}\begin{bmatrix} \frac{1-\nu}{1-2\nu} - M\bar{s}_{11}^2 & \frac{\nu}{1-2\nu} - M\bar{s}_{11}\bar{s}_{22} & \frac{\nu}{1-2\nu} - M\bar{s}_{11}\bar{s}_{33} & -M\bar{s}_{11}\bar{s}_{12} & -M\bar{s}_{11}\bar{s}_{13} & -M\bar{s}_{11}\bar{s}_{23} \\ & \frac{1-\nu}{1-2\nu} - M\bar{s}_{22}^2 & \frac{\nu}{1-2\nu} - M\bar{s}_{22}\bar{s}_{33} & -M\bar{s}_{22}\bar{s}_{12} & -M\bar{s}_{22}\bar{s}_{13} & -M\bar{s}_{22}\bar{s}_{23} \\ & & \frac{1-\nu}{1-2\nu} - M\bar{s}_{33}^2 & -M\bar{s}_{33}\bar{s}_{12} & -M\bar{s}_{33}\bar{s}_{13} & -M\bar{s}_{33}\bar{s}_{23} \\ & & & \frac{1}{2} - M\bar{s}_{12}^2 & -M\bar{s}_{12}\bar{s}_{13} & -M\bar{s}_{12}\bar{s}_{23} \\ & & & & \frac{1}{2} - M\bar{s}_{13}^2 & -M\bar{s}_{13}\bar{s}_{23} \\ & & & & & \frac{1}{2} - M\bar{s}_{23}^2 \end{bmatrix}$$

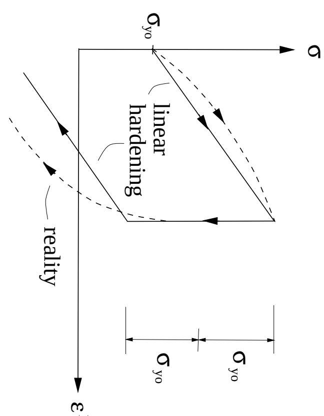
$$M = \frac{9G}{2A\sigma_{yo}^2} \qquad A = H + 3G$$

KINEMATIC HARDENING OF VON MISES MATERIAL

Nonlinear hardening $c = c(\epsilon^p_{eff})$

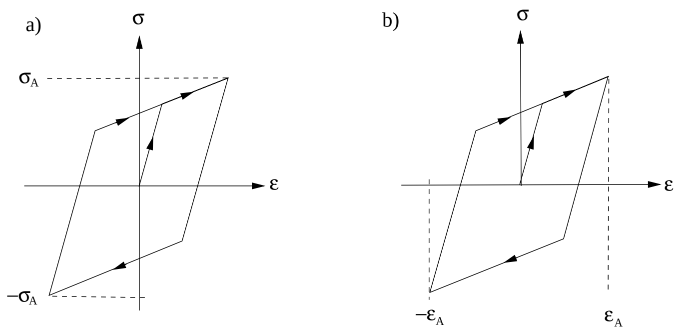


Linear hardening, H=constant, i.e. c=constant

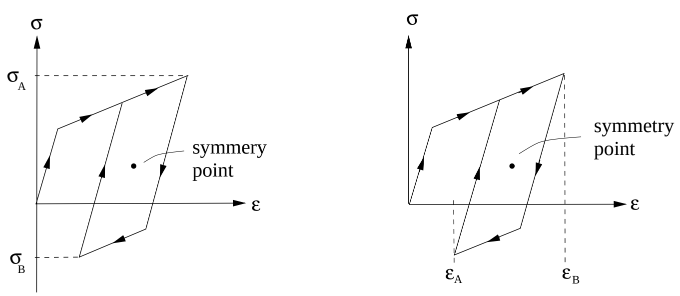


KINEMATIC HARDENING OF VON MISES MATERIAL

Symmetric cyclic loading

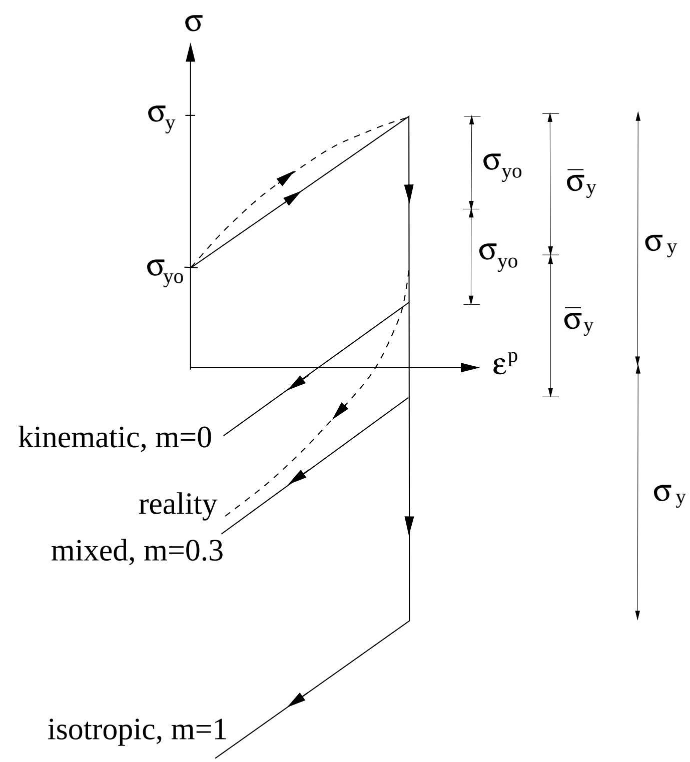


Unsymmetric cyclic loading



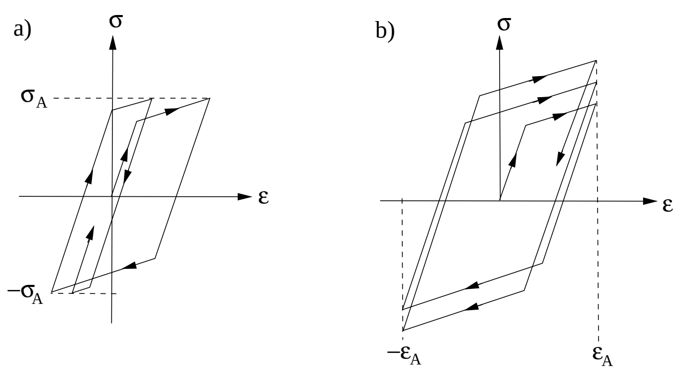
MIXED HARDENING OF VON MISES MATERIAL

Linear hardening

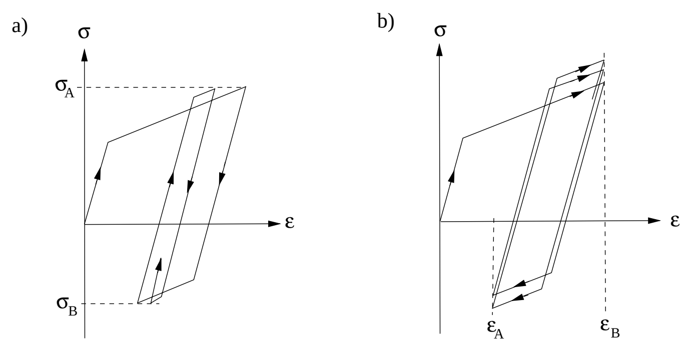


MIXED HARDENING OF VON MISES MATERIAL

Symmetric cyclic loading

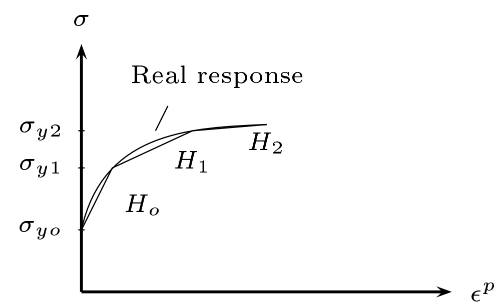


Unsymmetric cyclic loading

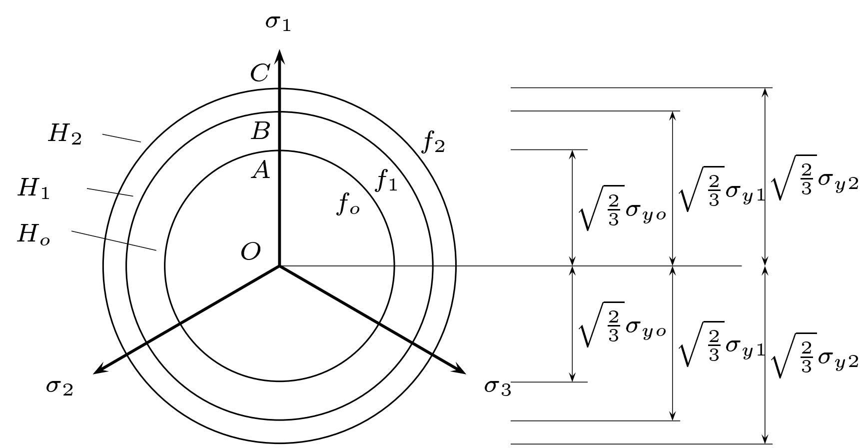


THE MRÓZ MODEL

Multilinear approximation of uniaxial response

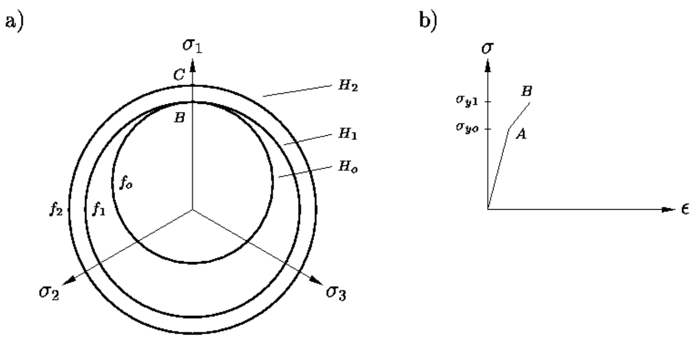


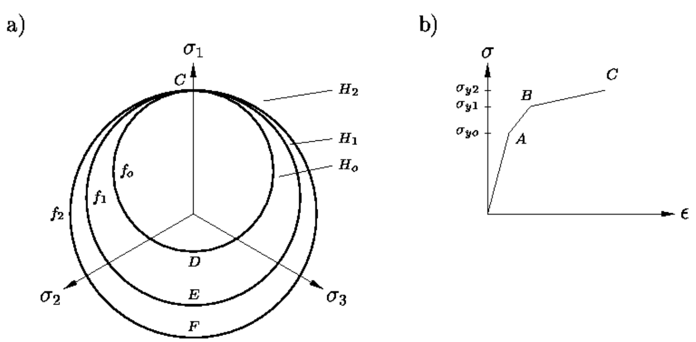
Position of von Mises surfaces



THE MRÓZ MODEL

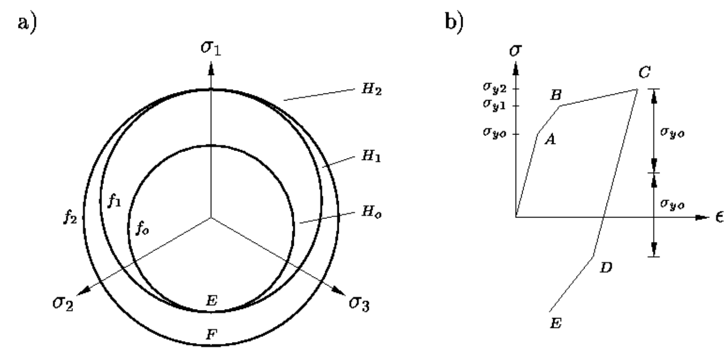
Increasing unaxial loading

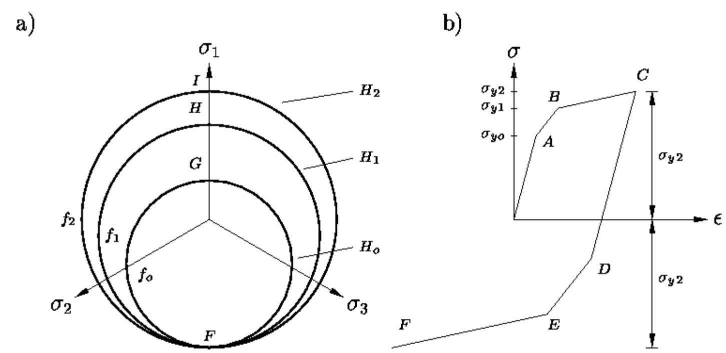




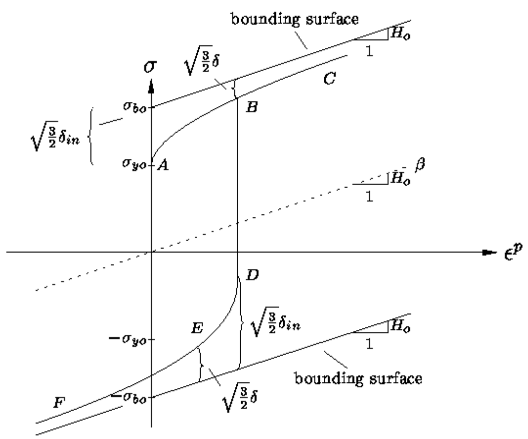
THE MRÓZ MODEL

Reversed unaxial loading

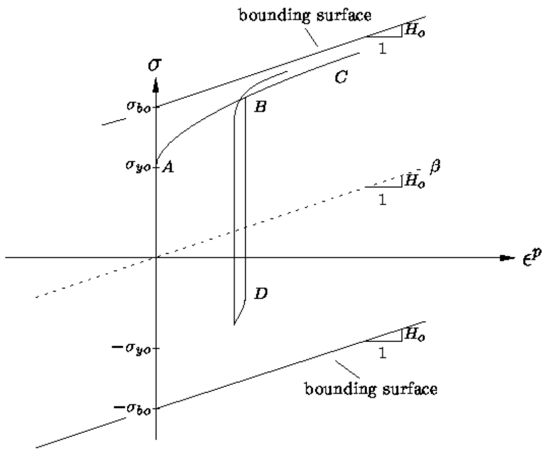




BOUNDING SURFACE MODELS



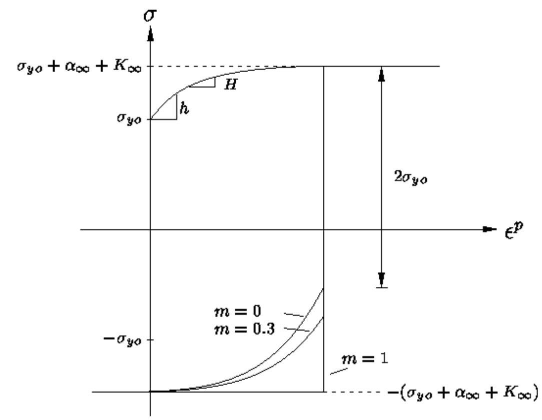
Overshooting effect



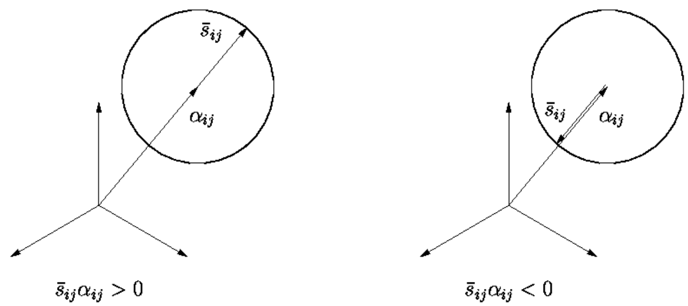
ARMSTRONG-FREDRICK'S MODELS

– Mixed hardening –

Prediction of mixed hardening

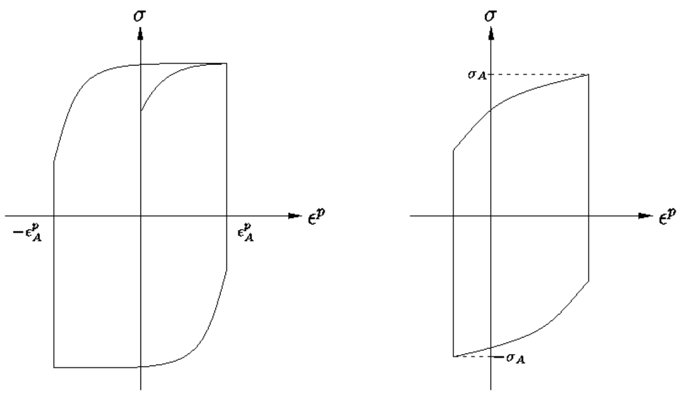


Load reversal

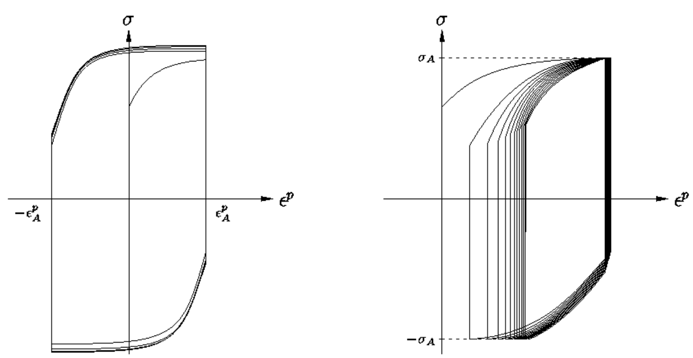


ARMSTRONG-FREDRICK'S MODELS

Symmetric cyclic loading, pure kinematic hardening

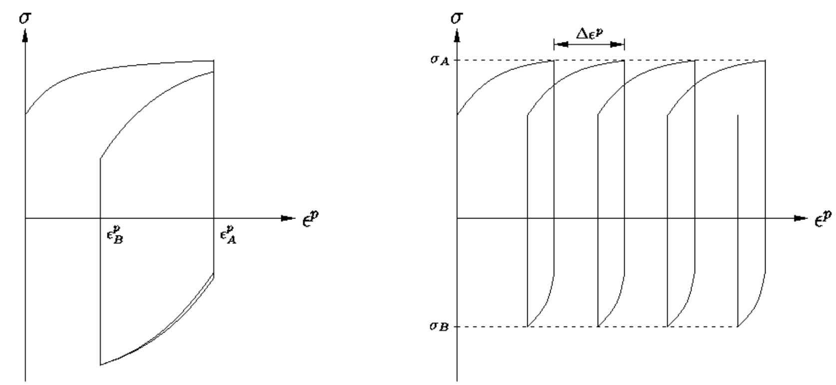


Symmetric cyclic loading, mixed hardening



ARMSTRONG-FREDRICK'S MODELS

Kinematic hardening unsymmetric cyclic loading



ARMSTRONG-FREDRICK'S MODEL
–von Mises, nonlinear kinematic hardening–

Yield function (assuming plasticity)

$$f = \left(\frac{3}{2}\bar{s}_{ij}\bar{s}_{ij}\right)^{1/2} - \sigma_{yo} = 0 \qquad \bar{s}_{ij} = s_{ij} - \alpha_{ij}$$

evolution law of A-F

$$\dot{\alpha}_{ij} = h(\frac{2}{3}\dot{\epsilon}^p_{ij} - \frac{\alpha_{ij}}{\alpha_\infty}\dot{\epsilon}^p_{eff})$$

Flow rule

$$\dot{\epsilon}^p_{ij} = \dot{\lambda}\frac{\partial f}{\partial \sigma_{ij}} = \dot{\lambda}\frac{3}{2}\frac{\bar{s}_{ij}}{\sigma_{yo}}$$

Effective plastic strain rate

$$\dot{\epsilon}^p_{eff} = \left(\frac{2}{3}\dot{\epsilon}^p_{ij}\dot{\epsilon}^p_{ij}\right)^{1/2} = \dot{\lambda}$$

Generalized plastic modulus

$$\dot{f} = 0 \quad \Rightarrow \quad \frac{\partial f}{\partial \sigma_{ij}}\dot{\sigma}_{ij} + \frac{\partial f}{\partial \alpha_{ij}}\dot{\alpha}_{ij} = 0$$

$$\frac{\partial f}{\partial \sigma_{ij}}\dot{\sigma}_{ij} + \frac{\partial f}{\partial \alpha_{ij}}h(\frac{2}{3}\dot{\epsilon}^p_{ij} - \frac{\alpha_{ij}}{\alpha_\infty}\dot{\epsilon}^p_{eff}) = 0$$

$$\frac{\partial f}{\partial \sigma_{ij}}\dot{\sigma}_{ij} + \dot{\lambda}\underbrace{[\frac{\partial f}{\partial \alpha_{ij}}h(\frac{\bar{s}_{ij}}{\sigma_{yo}} - \frac{\alpha_{ij}}{\alpha_\infty})]}_{-H} = 0$$

ARMSTRONG-FREDRICK'S MODEL
–von Mises, nonlinear kinematic hardening–

We found

$$H = -\frac{\partial f}{\partial \alpha_{ij}}h(\frac{\bar{s}_{ij}}{\sigma_{yo}} - \frac{\alpha_{ij}}{\alpha_\infty})$$

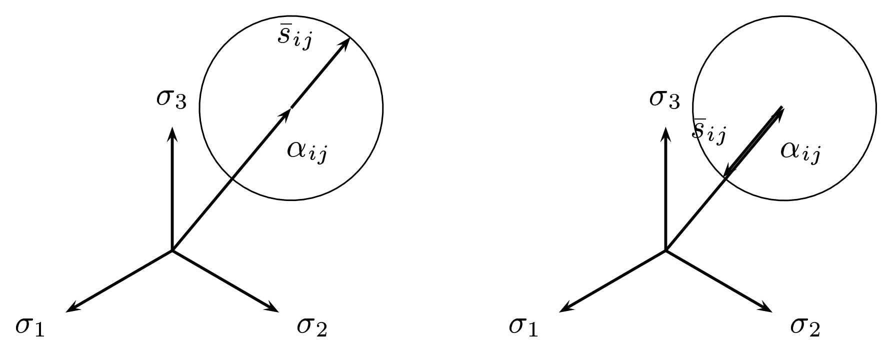
where

$$\frac{\partial f}{\partial \alpha_{ij}} = -\frac{3}{2}\frac{\bar{s}_{ij}}{\sigma_{yo}}$$

i.e.

$$\boxed{H = h(1 - \frac{3}{2}\frac{\bar{s}_{ij}\alpha_{ij}}{\sigma_{yo}\alpha_\infty})}$$

Generalized plastic modulus is not constant, different values depending on load direction



$\bar{s}_{ij}\alpha_{ij} > 0$ $\qquad$ $\bar{s}_{ij}\alpha_{ij} < 0$

Div. of Solid Mechanics, Lund University

ELASTO-PLASTIC STIFFNESS TENSOR –CORRESPONDING MATRIX FORMAT–

Yield function

$$f = f(\sigma_{ij}, \underset{\substack{\uparrow \\ \text{hardening parameters}}}{K^\alpha})$$

Kinematics

$$\epsilon_{ij} = \epsilon_{ij}^e + \epsilon_{ij}^p$$

Hooke's law $\sigma_{ij} = D_{ijkl}\epsilon_{kl}^e$, D_{ijkl} is constant, i.e.

$$\dot{\sigma}_{ij} = D_{ijkl}(\dot{\epsilon}_{kl} - \dot{\epsilon}_{kl}^p)$$

Flow rule

$$\dot{\epsilon}_{ij} = \dot{\lambda}\frac{\partial g}{\partial \sigma_{ij}} \qquad g = g(\sigma_{ij}, K^\alpha)$$

i.e.

$$\dot{\sigma}_{ij} = D_{ijkl}\dot{\epsilon}_{kl} - \dot{\lambda}D_{ijst}\frac{\partial g}{\partial \sigma_{st}}$$

Consistency $\dot{f} = 0$ during plastic loading, i.e.

$$\dot{f} = \frac{\partial f}{\partial \sigma_{ij}}\dot{\sigma}_{ij} + \frac{\partial f}{\partial K^\alpha}\dot{K}^\alpha = 0$$

ELASTO-PLASTIC STIFFNESS TENSOR –CORRESPONDING MATRIX FORMAT–

We found

$$\dot{\sigma}_{ij} = D_{ijkl}\dot{\epsilon}_{kl} - \dot{\lambda}D_{ijst}\frac{\partial g}{\partial \sigma_{st}} \tag{1}$$

and

$$\dot{f} = \frac{\partial f}{\partial \sigma_{ij}}\dot{\sigma}_{ij} + \frac{\partial f}{\partial K^\alpha}\dot{K}^\alpha = 0 \tag{2}$$

Moreover

$$K^\alpha = K^\alpha(\underset{\substack{\uparrow \\ \text{internal variables}}}{\kappa^\beta}), \quad \text{i.e.} \quad \dot{K}^\alpha = \frac{\partial K^\alpha}{\partial \kappa^\beta}\dot{\kappa}^\beta \tag{3}$$

Evolution law for $\dot{\kappa}$

$$\dot{\kappa}^\beta = \dot{\lambda}\,\underbrace{k^\beta(\sigma_{ij}, K^\alpha)}_{\text{evolution function (that we choose}}$$

Insertion in (3)

$$\dot{K}^\alpha = \dot{\lambda}\frac{\partial K^\alpha}{\partial \kappa^\beta}k^\beta$$

into (2)

$$\frac{\partial f}{\partial \sigma_{ij}}\dot{\sigma}_{ij} + \frac{\partial f}{\partial K^\alpha}\frac{\partial K^\alpha}{\partial \kappa^\beta}k^\beta\dot{\lambda} = 0$$

ELASTO-PLASTIC STIFFNESS TENSOR –CORRESPONDING MATRIX FORMAT–

We found

$$\dot{\sigma}_{ij} = D_{ijkl}\dot{\epsilon}_{kl} - \dot{\lambda} D_{ijst}\frac{\partial g}{\partial \sigma_{st}} \qquad (1)$$

and

$$\frac{\partial f}{\partial \sigma_{ij}}\dot{\sigma}_{ij} + \frac{\partial f}{\partial K^\alpha}\frac{\partial K^\alpha}{\partial \kappa^\beta}k^\beta \dot{\lambda} = 0$$

Define the generalized plastic modulus

$$H = -\frac{\partial f}{\partial K^\alpha}\frac{\partial K^\alpha}{\partial \kappa^\beta}k^\beta$$

then

$$\frac{\partial f}{\partial \sigma_{ij}}\dot{\sigma}_{ij} - H\dot{\lambda} = 0$$

Using (1) yields

$$\frac{\partial f}{\partial \sigma_{ij}}D_{ijkl}\dot{\epsilon}_{kl} - \dot{\lambda}(\frac{\partial f}{\partial \sigma_{ij}}D_{ijst}\frac{\partial g}{\partial \sigma_{st}} + H) = 0$$

where

$$A = \frac{\partial f}{\partial \sigma_{ij}}D_{ijst}\frac{\partial g}{\partial \sigma_{st}} + H > 0$$

i.e.

$$\dot{\lambda} = \frac{1}{A}\frac{\partial f}{\partial \sigma_{ij}}D_{ijkl}\dot{\epsilon}_{kl}$$

ELASTO-PLASTIC STIFFNESS TENSOR –CORRESPONDING MATRIX FORMAT–

We found

$$\dot{\sigma}_{ij} = D_{ijkl}\dot{\epsilon}_{kl} - \dot{\lambda} D_{ijst}\frac{\partial g}{\partial \sigma_{st}}$$

and

$$\dot{\lambda} = \frac{1}{A}\frac{\partial f}{\partial \sigma_{ij}}D_{ijkl}\dot{\epsilon}_{kl}$$

In conclusion (strain driven format)

$$\boxed{\dot{\sigma}_{ij} = D^{ep}_{ijkl}\dot{\epsilon}_{kl}}$$

where

$$\boxed{D^{ep}_{ijkl} = D_{ijkl} - \frac{1}{A}D_{ijst}\frac{\partial g}{\partial \sigma_{st}}\frac{\partial f}{\partial \sigma_{mn}}D_{mnkl}}$$

where

$$A = \frac{\partial f}{\partial \sigma_{ij}}D_{ijst}\frac{\partial g}{\partial \sigma_{st}} + H > 0$$

and

$$H = -\frac{\partial f}{\partial K^\alpha}\frac{\partial K^\alpha}{\partial \kappa^\beta}k^\beta$$

ELASTO-PLASTIC STIFFNESS TENSOR –CORRESPONDING MATRIX FORMAT–

General remarks

$$\dot{\sigma}_{ij} = D^{ep}_{ijkl}\dot{\epsilon}_{kl}$$

where

$$D^{ep}_{ijkl} = D_{ijkl} - \frac{1}{A}D_{ijst}\frac{\partial g}{\partial \sigma_{st}}\frac{\partial f}{\partial \sigma_{mn}}D_{mnkl}$$

where

$$A = \frac{\partial f}{\partial \sigma_{ij}}D_{ijst}\frac{\partial g}{\partial \sigma_{st}} + H, \qquad H = -\frac{\partial f}{\partial K^{\alpha}}\frac{\partial K^{\alpha}}{\partial \kappa^{\beta}}k^{\beta}$$

Having chosen f and g, it is the quantity H that is of importance

Route 1:

Choose $K^{\alpha} = K^{\alpha}(\kappa^{\beta})$, i.e. $\dot{K}^{\alpha} = \frac{\partial K^{\alpha}}{\partial \kappa^{\beta}}\dot{\kappa}^{\beta}$

Choose $\dot{\kappa}^{\beta} = \dot{\lambda}k^{\beta}$

i.e. $\dot{K}^{\beta} = \dot{\lambda}\frac{\partial K^{\alpha}}{\partial \kappa^{\beta}}k^{\beta}$

Route 2:

Choose directly $\dot{K}^{\beta} = \dot{\lambda}\frac{\partial K^{\alpha}}{\partial \kappa^{\beta}}k^{\beta} = \dot{\lambda}\text{function}$

ELASTO-PLASTIC STIFFNESS TENSOR –CORRESPONDING MATRIX FORMAT–

Elasticity

$$\sigma_{ij} = D_{ijkl}\epsilon_{kl}$$

Matrix format

$$\boldsymbol{\sigma} = \boldsymbol{D}\boldsymbol{\epsilon}$$

where

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{bmatrix} \quad \boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{12} \\ 2\epsilon_{13} \\ 2\epsilon_{23} \end{bmatrix} \quad \boldsymbol{D} = \begin{bmatrix} D_{11} & D_{12} & \ldots & D_{16} \\ D_{21} & D_{22} & \ldots & D_{26} \\ \vdots & & & \\ D_{61} & D_{62} & \ldots & D_{66} \end{bmatrix}$$

If the *tensor* equation (elasto-plasticity)

$$\dot{\sigma}_{ij} = D^{ep}_{ijkl}\dot{\epsilon}_{kl}$$

then in a completely similar manner we obtain

$$\dot{\boldsymbol{\sigma}} = \boldsymbol{D}^{ep}\dot{\boldsymbol{\epsilon}}$$

What happens if f and g are not expressed in σ_{ij} but in $\boldsymbol{\sigma}$? (the case with the classical anisotropic von Mises case, $f = \boldsymbol{\sigma}^T\boldsymbol{P}\boldsymbol{\sigma} - 1$)

ELASTO-PLASTIC STIFFNESS TENSOR
–CORRESPONDING MATRIX FORMAT–

In tensor notation

$$\dot{\epsilon}_{ij}^p = \dot{\lambda}\frac{\partial f}{\partial \sigma_{ij}}$$

In matrix notation

$$\dot{\boldsymbol{\epsilon}}^p = \dot{\lambda}\frac{\partial f}{\partial \boldsymbol{\sigma}}$$

where –as usual–

$$\boldsymbol{\epsilon}^p = \begin{bmatrix} \epsilon_{11}^p \\ \epsilon_{22}^p \\ \epsilon_{33}^p \\ 2\epsilon_{12}^p \\ 2\epsilon_{13}^p \\ 2\epsilon_{23}^p \end{bmatrix}$$

What do we mean by $\dfrac{\partial f}{\partial \boldsymbol{\sigma}}$?

We have that

$$\dot{\epsilon}_{12}^p = \dot{\lambda}\frac{\partial f}{\partial \sigma_{12}} \quad \text{and} \quad \dot{\epsilon}_{21}^p = \dot{\lambda}\frac{\partial f}{\partial \sigma_{21}}$$

Taking advantage of the symmetry properties we find

$$2\dot{\epsilon}_{12}^p = \dot{\lambda}(\frac{\partial f}{\partial \sigma_{12}} + \frac{\partial f}{\partial \sigma_{21}})$$

ELASTO-PLASTIC STIFFNESS TENSOR
–CORRESPONDING MATRIX FORMAT–

Using symmetry

$$2\dot{\epsilon}_{12}^p = \dot{\lambda}(\frac{\partial f}{\partial \sigma_{12}} + \frac{\partial f}{\partial \sigma_{21}})$$

If advantage is take of the symmetry of the stress tensor, we do not –for instance– differ between σ_{12} and σ_{21}, we treat them as the same quantity, i.e.

$$2\dot{\epsilon}_{12}^p = \dot{\lambda}\frac{\partial \hat{f}}{\partial \sigma_{12}}$$

Let us then define

$$\boxed{\frac{\partial \hat{f}}{\partial \boldsymbol{\sigma}} = \begin{bmatrix} \dfrac{\partial \hat{f}}{\partial \sigma_{11}} \\ \dfrac{\partial \hat{f}}{\partial \sigma_{22}} \\ \dfrac{\partial \hat{f}}{\partial \sigma_{33}} \\ \dfrac{\partial \hat{f}}{\partial \sigma_{12}} \\ \dfrac{\partial \hat{f}}{\partial \sigma_{13}} \\ \dfrac{\partial \hat{f}}{\partial \sigma_{23}} \end{bmatrix} \quad \text{and} \quad \dot{\boldsymbol{\epsilon}}^p = \dot{\lambda}\frac{\partial \hat{f}}{\partial \boldsymbol{\sigma}}}$$

Example: usual von Mises isotropic hardening

$$f = \left(\frac{3}{2} s_{ij} s_{ij}\right)^{1/2} - \sigma_y$$

Written explicitly

$$f = \left(\frac{3}{2}(s_{11}^2 + s_{22}^2 + s_{33}^2 + s_{12}^2 + s_{21}^2 + s_{13}^2 + s_{31}^2 + s_{23}^2 + s_{32}^2)\right)^{1/2} - \sigma_y$$

for instance

$$\dot{\epsilon}_{12}^p = \dot{\lambda} \frac{\partial f}{\partial \sigma_{12}} = \frac{3 s_{12}}{2 \sigma_y}$$

If advantage is taken of the symmetry of the stress tensor, then

$$\hat{f} = \left(\frac{3}{2}(s_{11}^2 + s_{22}^2 + s_{33}^2 + 2s_{12}^2 + 2s_{13}^2 + 2s_{23}^2)\right)^{1/2} - \sigma_y$$

i.e.

$$2\dot{\epsilon}_{12}^p = \dot{\lambda} \frac{\partial \hat{f}}{\partial \sigma_{12}} = \frac{3 s_{12}}{\sigma_y}$$

In conclusion, the case

$$\dot{\sigma}_{ij} = D_{ijkl} \dot{\epsilon}_{kl}$$

where

$$D_{ijkl}^{ep} = D_{ijkl} - \frac{1}{A} D_{ijst} \frac{\partial g}{\partial \sigma_{st}} \frac{\partial f}{\partial \sigma_{mn}} D_{mnkl}$$

is equivalent with

$$\boxed{\dot{\boldsymbol{\sigma}} = \boldsymbol{D}^{ep} \dot{\boldsymbol{\epsilon}}}$$

where

$$\boxed{\boldsymbol{D}^{ep} = \boldsymbol{D} - \frac{1}{A} \boldsymbol{D} \frac{\partial \hat{g}}{\partial \boldsymbol{\sigma}} \left(\frac{\partial \hat{f}}{\partial \boldsymbol{\sigma}}\right)^T \boldsymbol{D}}$$

and

$$A = \left(\frac{\partial \hat{f}}{\partial \boldsymbol{\sigma}}\right)^T \boldsymbol{D} \frac{\partial \hat{g}}{\partial \boldsymbol{\sigma}} + H$$

WEAK FORM OF EQUATIONS OF MOTION –PRINCIPLE OF VIRTUAL WORK–

Divergence theorem

$$\int_V c_{j,j} dV = \int_S c_j n_j dS$$

Equations of motion

$$\boxed{\sigma_{ij,j} + b_i = \rho \ddot{u}_i}$$

Multiply by arbitrary weight vector v_i and integrate

$$\int_V v_i \sigma_{ij,j} dV + \int_V v_i b_i dV = \int_V \rho \ddot{u}_i dV$$

Note that $v_i \sigma_{ij,j} = (v_i \sigma_{ij})_{,j} - v_{i,j} \sigma_{ij}$

$$\int_V v_i \sigma_{ij,j} dV = \underbrace{\int_V (v_i \sigma_{ij})_{j} dV}_{\int_S v_i \underbrace{\sigma_{ij} n_j}_{t_i} dS} - \int_V v_{i,j} \sigma_{ij} dV$$

$$\int_S v_i t_i dS - \int_V v_{i,j} \sigma_{ij} dV + \int_V v_i b_i dV = \int_V \rho v_i \ddot{u}_i dV$$

WEAK FORM OF EQUATIONS OF MOTION –PRINCIPLE OF VIRTUAL WORK–

$$\int_V \rho v_i \ddot{u}_i dV + \int_V v_{i,j} \sigma_{ij} dV = \underbrace{\int_S v_i t_i dS + \int_V v_i b_i dV}_{\text{external "virtual" work}}$$

holds for all materials

Define

$$\epsilon_{ij}^v = \frac{1}{2}(v_{i,j} + v_{j,i})$$

$$\Rightarrow \quad v_{i,j} \sigma_{ij} = \epsilon_{ij}^v \sigma_{ij}$$

$$\int_V \rho v_i \ddot{u}_i dV \int_V \epsilon_{ij} \sigma_{ij} dV = \int_S v_i t_i dS + \int_V v_i b_i dV$$

Define the matrices

$$\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \quad \boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \quad \boldsymbol{t} = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} \quad \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \quad \boldsymbol{\epsilon}^v = \begin{bmatrix} \epsilon_{11}^v \\ \epsilon_{22}^v \\ \epsilon_{33}^v \\ 2\epsilon_{12}^v \\ 2\epsilon_{13}^v \\ 2\epsilon_{23}^v \end{bmatrix} \quad \boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11}^v \\ \sigma_{22}^v \\ \sigma_{33}^v \\ \sigma_{12}^v \\ \sigma_{13}^v \\ \sigma_{23}^v \end{bmatrix}$$

$$\boxed{\int_V \rho \boldsymbol{v}^T \ddot{\boldsymbol{u}} dV + \int_V (\boldsymbol{\epsilon}^v)^T \boldsymbol{\sigma} dV = \int_S \boldsymbol{v}^T \boldsymbol{t} dS + \int_V \boldsymbol{v}^T \boldsymbol{b} dV}$$

FINITE ELEMENT FORMULATION

We found

$$\int_V \rho \boldsymbol{v}^T \ddot{\boldsymbol{u}} dV + \int_V (\boldsymbol{\epsilon}^v)^T \boldsymbol{\sigma} dV = \int_S \boldsymbol{v}^T \boldsymbol{t} dS + \int_V \boldsymbol{v}^T \boldsymbol{b} dV$$

FE-approximation

$$\boxed{\boldsymbol{u}(x_k, t) = \boldsymbol{N}(x_k)\boldsymbol{a}(t) \Rightarrow \boldsymbol{\epsilon} = \boldsymbol{B}\boldsymbol{a}}$$

Galerkin approach

$$\boldsymbol{v} = \boldsymbol{N}\boldsymbol{c} \qquad \Rightarrow \qquad \boldsymbol{\epsilon}^v = \boldsymbol{B}\boldsymbol{c}$$

where $\boldsymbol{c}$ is arbitrary and does not depend on position

$$\underbrace{}_{\text{itrary}} \Big[\int_V \rho \boldsymbol{N}^T \ddot{\boldsymbol{u}} dV + \int_V \boldsymbol{B}^T \boldsymbol{\sigma} dV - \int_S \boldsymbol{N}^T \boldsymbol{t} dS - \int_V \boldsymbol{N}^T \boldsymbol{b} dV \Big] = 0$$

$$\int_V \rho \boldsymbol{N}^T \ddot{\boldsymbol{u}} dV + \int_V \boldsymbol{B}^T \boldsymbol{\sigma} dV = \underbrace{\int_S \boldsymbol{N}^T \boldsymbol{t} dS + \int_V \boldsymbol{N}^T \boldsymbol{b} dV}_{= f \text{ ext. forces}}$$

Inertia term

$$\ddot{\boldsymbol{u}} = \boldsymbol{N}\ddot{\boldsymbol{a}} \quad \Rightarrow \quad \underbrace{\int_V \rho \boldsymbol{N}^T \boldsymbol{N} dV}_{\text{mass matrix}} \ddot{\boldsymbol{a}} = \boldsymbol{M}\ddot{\boldsymbol{a}}$$

$$\boxed{\boldsymbol{M}\ddot{\boldsymbol{a}} + \int_V \boldsymbol{B}^T \boldsymbol{\sigma} dV = f}$$

holds for all materials

FINITE ELEMENT FORMULATION

We found

$$\boldsymbol{M}\ddot{\boldsymbol{a}} + \int_V \boldsymbol{B}^T \boldsymbol{\sigma} dV = \boldsymbol{f}$$

where

$$\boldsymbol{M} = \int_V \rho \boldsymbol{N}^T \boldsymbol{N} dV \qquad \boldsymbol{f} = \int_S \boldsymbol{N}^T \boldsymbol{t} dS + \int_V \boldsymbol{N}^T \boldsymbol{b} dV$$

Static problems $\ddot{\boldsymbol{a}} = 0$

$$\boldsymbol{\psi} = 0 \quad \text{equilibrium equations}$$

where

$$\boldsymbol{\psi} = \int_V \boldsymbol{B}^T \boldsymbol{\sigma} dV - \boldsymbol{f}$$

This is a global problem

Integration along load path of

$$\dot{\boldsymbol{\sigma}} = \boldsymbol{D}^{ep} \dot{\boldsymbol{\epsilon}}$$

This is a local problem (should be solved at each material point irrespective of what happens in neighbouring points)

FULL NEWTON-RAPHSON SCHEME

· *Initiation of quantities*

$\boldsymbol{a}_0$; $\boldsymbol{\epsilon}_0$; $\boldsymbol{\sigma}_0$; $\boldsymbol{f}_0$; $\boldsymbol{f}_{int}$

· *For load step* $n = 0, 1, 2, \ldots N_{max}$

- · *Determine new load level* $\boldsymbol{f}_{n+1}$
- · *Initiation of iteration quantities*

  $\boldsymbol{a}^0 := \boldsymbol{a}_n$
- · *Iterate* $i = 1, 2, \ldots$ *until* $|\boldsymbol{\psi}|_{norm} = |\boldsymbol{f}_{n+1} - \boldsymbol{f}_{int}|_{norm} < tol$
  - · *Calculate* $\boldsymbol{K}_t = \int_V \boldsymbol{B}^T \boldsymbol{D}_t^i \boldsymbol{B} dV$
  - · *Calculate* $\boldsymbol{a}^i$ *from* $\boldsymbol{K}_t(\boldsymbol{a}^i - \boldsymbol{a}^{i-1}) = \boldsymbol{f}_{n+1} - \boldsymbol{f}_{int}$
  - · *Calculate* $\boldsymbol{\epsilon}^i := \boldsymbol{B}\boldsymbol{a}^i$
  - · *Determine* $\boldsymbol{\sigma}^i$ *by integration of the constitutive equations* (see next chapter)
  - · *Calculate internal forces* $\boldsymbol{f}_{int} = \int_V \boldsymbol{B}^T \boldsymbol{\sigma}^i dV$
- · *End iteration loop*
- · *Accept quantities*

  $\boldsymbol{a}_{n+1} := \boldsymbol{a}^i$; $\boldsymbol{\epsilon}_{n+1} := \boldsymbol{\epsilon}^i$; $\boldsymbol{\sigma}_{n+1} := \boldsymbol{\sigma}^i$; $\boldsymbol{f}_{int}$

· *End load step loop*

DYNAMIC CONSIDERATIONS
–discretization in time–

FE discretization, equations of motion

$$\boldsymbol{M}\ddot{\boldsymbol{a}} + \boldsymbol{\psi}(\boldsymbol{a}) = \boldsymbol{0}$$

where

$$\boldsymbol{M} = \int_V \rho \boldsymbol{N}^T \boldsymbol{N} dV \qquad \boldsymbol{psi}(\boldsymbol{a}) = \int_V \boldsymbol{B}^T \boldsymbol{\sigma} dV - \boldsymbol{f}$$

Task: Nonlinear diff. eqns. $\Rightarrow$ nonlinear algebraic eqns.

Newmark time integration scheme

$$\boxed{\begin{aligned} \boldsymbol{a}_{n+1} &= \boldsymbol{a}_n + \Delta t \dot{\boldsymbol{a}}_n + \tfrac{\Delta t^2}{2}[(1-2\beta)\ddot{\boldsymbol{a}}_n + 2\beta \ddot{\boldsymbol{a}}_{n+1}] \\ \dot{\boldsymbol{a}}_{n+1} &= \dot{\boldsymbol{a}}_n + \Delta t[(1-\gamma)\ddot{\boldsymbol{a}}_n + \gamma \ddot{\boldsymbol{a}}_{n+1}] \end{aligned}}$$

very general approximation, e.g.

$\beta = \frac{1}{4},\ \gamma = \frac{1}{2}$	$\Rightarrow$	trapezoidal rule
$\beta = 0,\ \gamma = \frac{1}{2}$	$\Rightarrow$	central diff. approximation
		(constant) average acceleration method
$\beta = \frac{1}{6},\ \gamma = \frac{1}{2}$	$\Rightarrow$	linear acceleration method
$\beta = \frac{1}{12},\ \gamma = \frac{1}{2}$	$\Rightarrow$	Fox-Godwin method
		royal road method

etc.

DYNAMIC CONSIDERATIONS

–Explicit scheme–

Assume that

$$\beta = 0, \qquad \gamma = \frac{1}{2}$$

From the Newmark scheme

$$\boldsymbol{a}_{n+1} = \boldsymbol{a}_n + \Delta t \dot{\boldsymbol{a}}_n + \tfrac{\Delta t^2}{2} \ddot{\boldsymbol{a}}_n$$
$$\dot{\boldsymbol{a}}_{n+1} = \dot{\boldsymbol{a}}_n + \tfrac{\Delta t}{2} (\ddot{\boldsymbol{a}}_n + \ddot{\boldsymbol{a}}_{n+1})$$

Solving for $\ddot{\boldsymbol{a}}_n$ yields

$$\boxed{\ddot{\boldsymbol{a}}_n = \tfrac{1}{\Delta t^2} (\boldsymbol{a}_{n+1} - 2\boldsymbol{a}_n + \boldsymbol{a}_{n-1})}$$

central difference approx. to $\ddot{\boldsymbol{a}}_n$

Equations of motion at the current time t_n

$$\boldsymbol{M} \ddot{\boldsymbol{a}}_n + \boldsymbol{\psi}(\boldsymbol{a}_n) = \boldsymbol{0}$$

or

$$\boxed{\boldsymbol{M} \boldsymbol{a}_{n+1} = \boldsymbol{M}(2\boldsymbol{a}_n - \boldsymbol{a}_{n-1}) + \Delta t^2 (\boldsymbol{f}_n - \int_V \boldsymbol{B}^T \boldsymbol{\sigma}_n dV)}$$

DYNAMIC CONSIDERATIONS

–Explicit scheme–

We found

$$\boldsymbol{M} \boldsymbol{a}_{n+1} = \boldsymbol{M}(2\boldsymbol{a}_n - \boldsymbol{a}_{n-1}) + \Delta t^2 (\boldsymbol{f}_n - \int_V \boldsymbol{B}^T \boldsymbol{\sigma}_n dV)$$

Used in <u>all</u> explicit FE-codes. LS-DYNA, Abaqus etc.

Assume that the mass matrix $\boldsymbol{M}$ is lumped, i.e.

$$\boldsymbol{M} = \begin{bmatrix} m_1 & & \\ & m_2 & \\ & & \ddots & \\ & & & m_{ndof} \end{bmatrix} \qquad \Rightarrow \text{diagonal}$$

No inversion of $\boldsymbol{M}$ is needed, the FE-system can be solve in a row by row fashion

Price to pay we must require that

$$\boxed{\Delta t \leq \frac{T_s}{\pi} \Rightarrow \quad \text{Stability}}$$

DYNAMIC CONSIDERATIONS
–Implicit scheme–

From the Newmark scheme (assume $\beta \neq 0$)

$$\ddot{\boldsymbol{a}}_{n+1} = \frac{1}{\beta \Delta t^2}(\boldsymbol{a}_{n+1} - \boldsymbol{a}_n) - \frac{1}{\beta \Delta t}\dot{\boldsymbol{a}}_n - \frac{1-2\beta}{2\beta}\ddot{\boldsymbol{a}}_n$$

Equations of motions at time t_{n+1}

$$\boldsymbol{M}\ddot{\boldsymbol{a}}_{n+1} + \boldsymbol{\psi}(\boldsymbol{a}_{n+1}) = \boldsymbol{0}$$

or

$$\underbrace{\boldsymbol{M}[\frac{1}{\beta \Delta t^2}(\boldsymbol{a}_{n+1} - \boldsymbol{a}_n) - \frac{1}{\beta \Delta t}\dot{\boldsymbol{a}}_n - \frac{1-2\beta}{2\beta}\ddot{\boldsymbol{a}}_n] + \boldsymbol{\psi}(\boldsymbol{a}_{n+1})}_{\boldsymbol{v}(\boldsymbol{a}_{n+1}) = \boldsymbol{0}} = \boldsymbol{0}$$

Transfor to standard iteration format

$$\boldsymbol{0} = -(\boldsymbol{A}(\boldsymbol{a}_{n+1}))^{-1}\boldsymbol{v}(\boldsymbol{a}_{n+1})$$

$$\boldsymbol{a}_{n+1} = \underbrace{\boldsymbol{a}_{n+1} - (\boldsymbol{A}(\boldsymbol{a}_{n+1}))^{-1}\boldsymbol{v}(\boldsymbol{a}_{n+1})}_{\boldsymbol{F}(\boldsymbol{a}_{n+1})}$$

Iteration scheme

$$\boldsymbol{a}_{n+1}^{i} = \boldsymbol{F}(\boldsymbol{a}_{n+1}^{i-1})$$

DYNAMIC CONSIDERATIONS
–Implicit scheme–

We obtained

$$\boxed{\begin{aligned} \boldsymbol{a}_{n+1}^{i} &= \boldsymbol{a}_{n+1}^{i-1} - (\boldsymbol{A}(\boldsymbol{a}_{n+1}^{i-1}))^{-1}\boldsymbol{v}(\boldsymbol{a}_{n+1}^{i-1}) \\ \boldsymbol{v}(\boldsymbol{a}_{n+1}^{i-1}) &= \boldsymbol{M}[\tfrac{1}{\beta \Delta t^2}(\boldsymbol{a}_{n+1}^{i-1} - \boldsymbol{a}_n) - \tfrac{1}{\beta \Delta t}\dot{\boldsymbol{a}}_n - \tfrac{1-2\beta}{2\beta}\ddot{\boldsymbol{a}}_n] \\ &\quad + \boldsymbol{\psi}(\boldsymbol{a}_{n+1}^{i-1}) \end{aligned}}$$

The Newton Raphson scheme

$$\boxed{\boldsymbol{A}^{i-1} = (\frac{1}{\beta \Delta t^2}\boldsymbol{M} + \boldsymbol{K}^{ep})^{i-1}}$$

Choice of parameters

$$\boxed{\gamma \geq 0,\ \beta \geq \frac{1}{4}(\gamma + \frac{1}{2}) \quad \Rightarrow \quad \text{unconditional stability}}$$

DYNAMIC CONSIDERATIONS

Newmark (1959) time integration scheme

$$\boldsymbol{a}_{n+1} = \boldsymbol{a}_n + \Delta t\dot{\boldsymbol{a}}_n + \tfrac{\Delta t^2}{2}[(1-2\beta)\ddot{\boldsymbol{a}}_n + 2\beta\ddot{\boldsymbol{a}}_{n+1}]$$
$$\dot{\boldsymbol{a}}_{n+1} = \dot{\boldsymbol{a}}_n + \Delta t[(1-\gamma)\ddot{\boldsymbol{a}}_n + \gamma\ddot{\boldsymbol{a}}_{n+1}]$$

Assume $\beta = \frac{1}{4} \qquad \gamma = \frac{1}{2}$

$$\boxed{\begin{aligned}\boldsymbol{a}_{n+1} &= \boldsymbol{a}_n + \Delta t\dot{\boldsymbol{a}}_n + \tfrac{\Delta t^2}{2}[\tfrac{1}{2}\ddot{\boldsymbol{a}}_n + \tfrac{1}{2}\beta\ddot{\boldsymbol{a}}_{n+1}]\\ \dot{\boldsymbol{a}}_{n+1} &= \dot{\boldsymbol{a}}_n + \tfrac{\Delta t}{2}[\ddot{\boldsymbol{a}}_n + \ddot{\boldsymbol{a}}_{n+1}]\end{aligned}}$$

Consider a different route, use the trapezoidal rule with $\theta = \frac{1}{2}$

$$\boldsymbol{a}_{n+1} = \boldsymbol{a}_n + \int_n^{n+1} \dot{\boldsymbol{a}}dt = \boldsymbol{a}_n + (\dot{\boldsymbol{a}})^*\Delta t$$

$$(\dot{\boldsymbol{a}})^* = (1-\theta)\dot{\boldsymbol{a}}_n + \theta\dot{\boldsymbol{a}}_{n+1} = \frac{1}{2}(\dot{\boldsymbol{a}}_n + \dot{\boldsymbol{a}}_{n+1})$$

i.e.

$$\boldsymbol{a}_{n+1} = \boldsymbol{a}_n + \tfrac{\Delta t}{2}(\dot{\boldsymbol{a}}_n + \dot{\boldsymbol{a}}_{n+1})$$

Likewise

$$\dot{\boldsymbol{a}}_{n+1} = \dot{\boldsymbol{a}}_n + \tfrac{\Delta t}{2}(\ddot{\boldsymbol{a}}_n + \ddot{\boldsymbol{a}}_{n+1})$$

use of $\boldsymbol{a}_{n+1}$

$$\boxed{\boldsymbol{a}_{n+1} = \boldsymbol{a}_n + \Delta t\dot{\boldsymbol{a}}_n + \frac{\Delta t^2}{2}[\frac{1}{2}\ddot{\boldsymbol{a}}_n + \frac{1}{2}\beta\ddot{\boldsymbol{a}}_{n+1}]}$$

FULL NEWTON-RAPHSON SCHEME

· *Initiation of quantities*

$\boldsymbol{a}_0$; $\dot{\boldsymbol{a}}_0$; $\ddot{\boldsymbol{a}}_0$; $\boldsymbol{\epsilon}_0$; $\boldsymbol{\sigma}_0$; $\boldsymbol{f}_0$; $\boldsymbol{f}_{int}$

· *For load step* $n = 0, 1, 2, ...N_{max}$

 · *Determine new load level* $\boldsymbol{f}_{n+1}$

 · *Initiation of iteration quantities*

 $\boldsymbol{a}^0 := \boldsymbol{a}_n$; $\dot{\boldsymbol{a}}^0 := \dot{\boldsymbol{a}}_n$; $\ddot{\boldsymbol{a}}^0 := \ddot{\boldsymbol{a}}_n$

 · *Iterate* $i = 1, 2, \ldots$ *until* $|\boldsymbol{v}|_{norm} < tol$

  · *Calculate* $\boldsymbol{A} = \dfrac{1}{\beta\Delta t^2}\boldsymbol{M} + \boldsymbol{K}_t$

  · *Calculate* $\boldsymbol{a}^i$ *from* $\boldsymbol{A}(\boldsymbol{a}^i - \boldsymbol{a}^{i-1}) = -\boldsymbol{v}$

  · *Calculate* $\boldsymbol{\epsilon}^i := \boldsymbol{B}\boldsymbol{a}^i$

  · *Determine* $\boldsymbol{\sigma}^i$

  · *Calculate* $\boldsymbol{\psi} = \boldsymbol{f}_{int} - \boldsymbol{f}_{n+1} = \int_V \boldsymbol{B}^T\boldsymbol{\sigma}^i dV - \boldsymbol{f}_{n+1}$

  · *Calculate*

  $$\boldsymbol{v} = \boldsymbol{M}[\tfrac{1}{\beta\Delta t^2}(\boldsymbol{a} - \boldsymbol{a}_n) - \tfrac{1}{\beta\Delta t}\dot{\boldsymbol{a}}_n - \tfrac{1-2\beta}{2\beta}\ddot{\boldsymbol{a}}_n] + \boldsymbol{\psi}$$

 · *End iteration loop*

 · *Accept quantities*

 $\boldsymbol{a}_{n+1} := \boldsymbol{a}^i$; $\dot{\boldsymbol{a}}_{n+1} := \dot{\boldsymbol{a}}^i$; $\ddot{\boldsymbol{a}}_{n+1} := \ddot{\boldsymbol{a}}^i$

 $\boldsymbol{\epsilon}_{n+1} := \boldsymbol{\epsilon}^i$; $\boldsymbol{\sigma}_{n+1} := \boldsymbol{\sigma}^i$; $\boldsymbol{f}_{int}$

· *End load step loop*

FUNDAMENTAL ASSUMPTIONS

Kinematics

$$\boxed{\dot{\epsilon}_{ij} = \dot{\epsilon}^e_{ij} + \dot{\epsilon}^p_{ij}}$$

Hooke $\sigma_{ij} = D_{ijkl}(\epsilon_{kl} - \epsilon^p_{kl})$ (D_{ijkl} constant) or

$$\boxed{\dot{\sigma}_{ij} = D_{ijkl}(\dot{\epsilon}_{kl} - \dot{\epsilon}_{kl})}$$

Yield Function

$$\boxed{f(\sigma_{ij}, K^\alpha) = 0} \quad \text{at plastic loading}$$

$$f(\sigma_{ij}, K^\alpha) < 0 \quad \text{elastic response}$$

Flow rule

$$\boxed{\dot{\epsilon}_{ij} = \dot{\lambda}\frac{\partial g}{\partial \sigma_{ij}}} \qquad \dot{\lambda} \geq 0 \qquad g = g(\sigma_{ij}, K^\alpha)$$

Consistency

$$\boxed{\dot{f} = \frac{\partial f}{\partial \sigma_{ij}}\dot{\sigma}_{ij} + \frac{\partial f}{\partial K^\alpha}\dot{K}^\alpha = 0}$$

Evolution laws

$$K^\alpha = K^\alpha(\kappa^\beta) \qquad \Rightarrow \qquad \boxed{\dot{K}^\alpha = \frac{\partial K^\alpha}{\partial \kappa^\beta}\dot{\kappa}^\beta}$$

$$\boxed{\dot{\kappa}^\beta = \dot{\lambda}k^\beta(\sigma_{ij}, K^\alpha)}$$

FULL NEWTON-RAPHSON SCHEME

- *Initiation of quantities*

  $\boldsymbol{a}_0$; $\boldsymbol{\epsilon}_0$; $\boldsymbol{\sigma}_0$; $\boldsymbol{f}_0$; $\boldsymbol{f}_{int}$
- *For load step* $n = 0, 1, 2, \ldots .N_{max}$
  - *Determine new load level* $\boldsymbol{f}_{n+1}$
  - *Initiation of iteration quantities*

    $\boldsymbol{a}^0 := \boldsymbol{a}_n$
  - *Iterate* $i = 1, 2, \ldots$ *until* $|\boldsymbol{\psi}|_{norm} = |\boldsymbol{f}_{n+1} - \boldsymbol{f}_{int}|_{norm} < tol$
    - *Calculate* $\boldsymbol{K}_t = \int_V \boldsymbol{B}^T \boldsymbol{D}^i_t \boldsymbol{B} dV$
    - *Calculate* $\boldsymbol{a}^i$ *from* $\boldsymbol{K}_t(\boldsymbol{a}^i - \boldsymbol{a}^{i-1}) = \boldsymbol{f}_{n+1} - \boldsymbol{f}_{int}$
    - *Calculate* $\boldsymbol{\epsilon}^i := \boldsymbol{B}\boldsymbol{a}^i$
    - *Determine* $\boldsymbol{\sigma}^i$ *by integration of the constitutive equations* (this chapter)
    - *Calculate internal forces* $\boldsymbol{f}_{int} = \int_V \boldsymbol{B}^T \boldsymbol{\sigma}^i dV$
  - *End iteration loop*
  - *Accept quantities*

    $\boldsymbol{a}_{n+1} := \boldsymbol{a}^i$; $\boldsymbol{\epsilon}_{n+1} := \boldsymbol{\epsilon}^i$; $\boldsymbol{\sigma}_{n+1} := \boldsymbol{\sigma}^i$; $\boldsymbol{f}_{int}$
- *End load step loop*

TASK OF THE CONSTITUTIVE DRIVER

GIVEN: A FINITE STRAIN INCREMENT

FIND: THE STRESS INCREMENT

Note that we assume a strain driven formulation

ELASTO-PLASTICITY

Elasto-plastic incremental relation

$$\dot{\sigma}_{ij} = D^{ep}_{ijkl}\dot{\epsilon}_{kl}$$

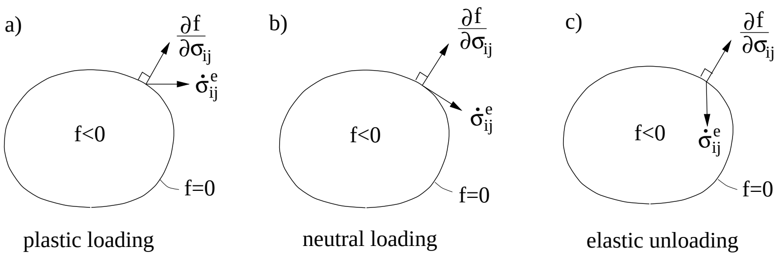
Integrate from 1 to 2, i.e.

$$\sigma^{(2)}_{ij} - \sigma^{(1)}_{ij} = \int_1^2 D^{ep}_{ijkl} d\epsilon_{kl}$$

Numerical integration is required

Does the loading imply elasto-plasticity?

$$\frac{\partial f}{\partial \sigma_{ij}} \underbrace{D_{ijkl}\dot{\epsilon}_{kl}}_{\dot{\sigma}^e_{ij}} \begin{cases} \geq 0 & \text{plast. loading} \\ < 0 & \text{elastic loading} \end{cases}$$



For plastic behaviour, the elastic stress increment tries to go outside the yield surface

DOES THE STEP IMPLY PLASTICITY

state 1: known $(\sigma_{ij},\ K^{\alpha(1)},\ \epsilon_{ij}^{p(1)},\ \epsilon_{ij}^{(1)})$

state 2: to be determined $(\sigma_{ij}^{(2)},\ K^{\alpha(2)},\ \epsilon_{ij}^{p(2)},\ \underbrace{\epsilon_{ij}^{(2)}}_{\text{known}}\ )$

Strain increment

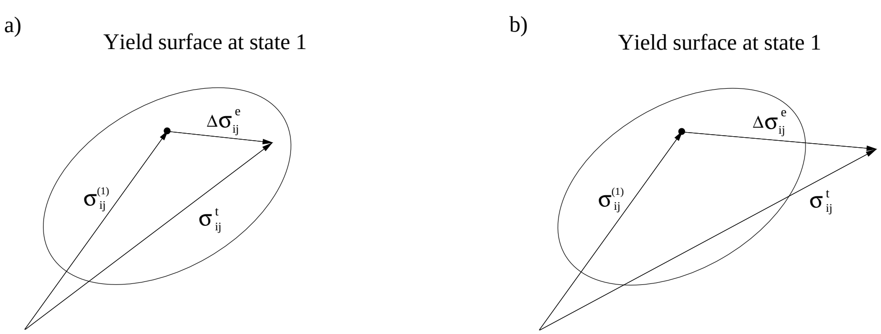
$$\Delta\epsilon_{ij} = \epsilon_{ij}^{(2)} - \epsilon_{ij}^{(1)} \qquad \text{known}$$

From Hooke's law $\dot{\epsilon}_{ij} = D_{ijkl}(\dot{\epsilon}_{kl} - \dot{\epsilon}_{kl})$ follows

$$\sigma_{ij}^{(2)} - \sigma_{ij}^{(1)} = \underbrace{D_{ijkl}\Delta\epsilon_{kl}}_{\Delta\sigma_{ij}^{e}} - D_{ijkl}\int_{\epsilon_{mn}^{p(1)}}^{\epsilon_{mn}^{p(2)}} d\epsilon_{kl}^{p}$$

Define trial stresses

$$\sigma_{ij}^{t} = \sigma_{ij}^{(1)} + \Delta\sigma_{ij}^{e} \quad \Rightarrow \quad \sigma_{ij}^{(2)} = \sigma_{ij}^{t} - D_{ijkl}\int_{(1)}^{(2)} d\epsilon_{kl}^{p}$$



LOADING AND UNLOADING CRITERIA

loading or unloading

$$f^{t} = f(\sigma_{ij}^{t}, K^{\alpha(1)})$$

If $\quad f^{t} > 0 \Rightarrow$ plastic
$\quad\; f^{t} \leq 0 \Rightarrow$ elastic

Program code

· *Calculate trial stresses*

$$\sigma_{ij}^{t} = \sigma_{ij}^{(1)} + D_{ijkl}\Delta\epsilon_{kl}$$

· *If* $f(\sigma^{t}, K^{\alpha(1)}) \leq 0$

Elastic response

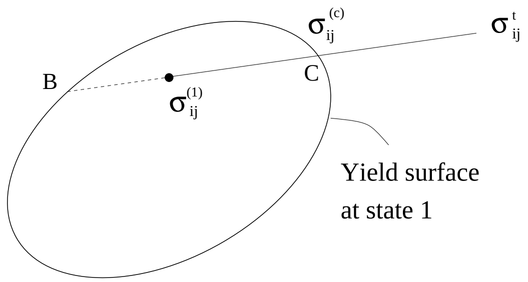
$$\sigma_{ij}^{(2)} = \sigma_{ij}^{t}$$

· *Else*

Elasto-plastic response

$$\sigma_{ij}^{(2)} = \sigma_{ij}^{t} - D_{ijkl}\int_{1}^{2} d\epsilon_{kl}^{p}$$

CONTACT STRESSES



Denote σ_{ij}^c as the contact stresses

Assuming linear interpolation of the strains

$$\epsilon_{ij}^c = (1-\gamma)\epsilon_{ij}^{(1)} + \gamma\epsilon_{ij}^{(2)}$$

gives linear interpolation of stresses (if D_{ijkl} constant), i.e.

$$\sigma_{ij}^c = (1-\gamma)\sigma_{ij}^{(1)} + \gamma\sigma_{ij}^{(2)}$$

Contact stresses must satisfy yield criterion

$$f^c = f(\sigma_{ij}^c, K^{\alpha(1)}) = 0$$

i.e.

$$f[(1-\gamma)\sigma_{ij}^{(1)} + \gamma\sigma_{ij}^{t}, K^{\alpha(1)}] = 0$$

One non-linear equation, one unknown γ

STRESS CALCULATION

– Indirect consideration to the consistency condition

numerically $\dot{f} = 0$ *is enforced*

– Direct consideration to the consistency condition

numerically $f = 0$ *is enforced*

DIRECT CONSIDERATION TO CONSISTENCY –RETURN METHODS–

Incr. form of Hooke's law $\dot{\sigma}_{ij} = D_{ijkl}(\dot{\epsilon}_{kl} - \dot{\epsilon}^p_{kl})$, integration

$$\sigma^{(2)}_{ij} - \sigma^{(1)}_{ij} = \underbrace{D_{ijkl}\Delta\epsilon_{kl}}_{\Delta\sigma^e_{ij}} - D_{ijkl}\int_1^2 d\epsilon^p_{kl}$$

Definition

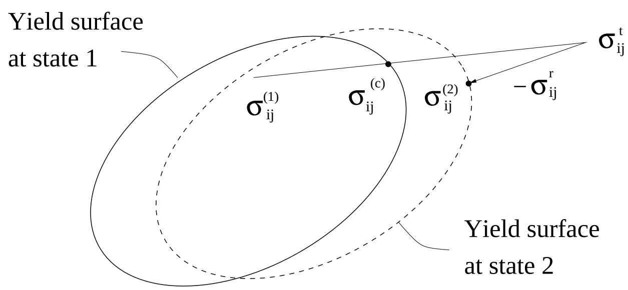
$$\sigma^t_{ij} = \sigma^{(1)}_{ij} + \Delta\sigma^e_{ij} \qquad \text{known}$$

Flow rule

$$d\epsilon^p_{kl} = d\lambda\frac{\partial g}{\partial\sigma_{kl}}$$

i.e.

$$\boxed{\sigma^{(2)}_{ij} = \sigma^t_{ij} - \sigma^r_{ij} \quad \text{where} \quad \sigma^r_{ij} = D_{ijkl}\int_c^2 \frac{\partial g}{\partial\sigma_{kl}} d\lambda}$$



DIRECT CONSIDERATION TO CONSISTENCY –RETURN METHODS–

We found

$$\sigma^{(2)}_{ij} = \sigma^t_{ij} - \sigma^r_{ij} \quad \text{where} \quad \sigma^r_{ij} = D_{ijkl}\int_c^2 \frac{\partial g}{\partial\sigma_{kl}} d\lambda$$

Approximate

$$\Delta\epsilon^p_{kl} = \int_c^2 \frac{\partial g}{\partial\sigma_{kl}} d\lambda \approx \left(\frac{\partial g}{\partial\sigma_{kl}}\right)^* \Delta\lambda$$

i.e.

$$\boxed{\sigma^{(2)}_{ij} = \sigma^t_{ij} - D_{ijkl}\left(\frac{\partial g}{\partial\sigma_{kl}}\right)^* \Delta\lambda}$$

Evolution laws

$$\dot{\kappa}^\alpha = \dot{\lambda}k^\alpha(\sigma_{kl}, K^\beta)$$

Approximate

$$\int_1^2 d\kappa^\alpha = \int_c^2 k^\alpha d\lambda \approx k^{\alpha(*)}\Delta\lambda$$

i.e.

$$\boxed{\kappa^{\alpha(2)} = \kappa^{\alpha(1)} + k^{\alpha(*)}\Delta\lambda} \qquad \boxed{K^{\alpha(2)} = K^\alpha(\kappa^{\beta(2)})}$$

Consistency

$$\boxed{f(\sigma^{(2)}_{ij}, K^{\alpha(2)}) = 0}$$

DIRECT CONSIDERATION TO CONSISTENCY –APPROXIMATIONS–

Illustration of generalized trapezoidal rule

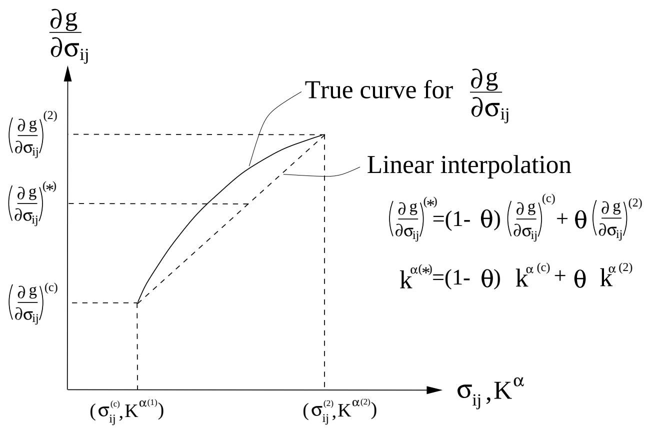
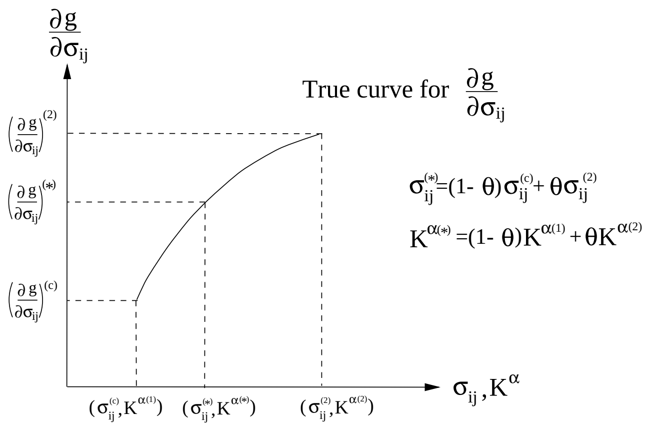


Illustration of generalized mid-point rule



DIRECT CONSIDERATION TO CONSISTENCY –Generalized mid-point rule–

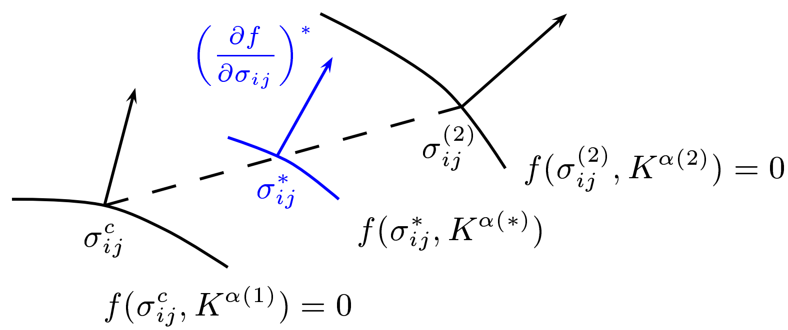
Assume associated plasticity

$$\int_c^2 \frac{\partial f}{\partial \sigma_{ij}} d\lambda \approx \Delta\lambda \left(\frac{\partial f}{\partial \sigma_{ij}}\right)^* = \Delta\lambda \left.\frac{\partial f}{\partial \sigma_{ij}}\right|_{(\sigma_{ij}^*, K^{\alpha(*)})}$$

where

$$\sigma_{ij}^* = (1-\theta)\sigma_{ij}^c + \theta\sigma_{ij}^{(2)}$$
$$K^* = (1-\theta)K^{(1)} + \theta K^{(2)}$$

Stress space



DIRECT CONSIDERATION TO CONSISTENCY

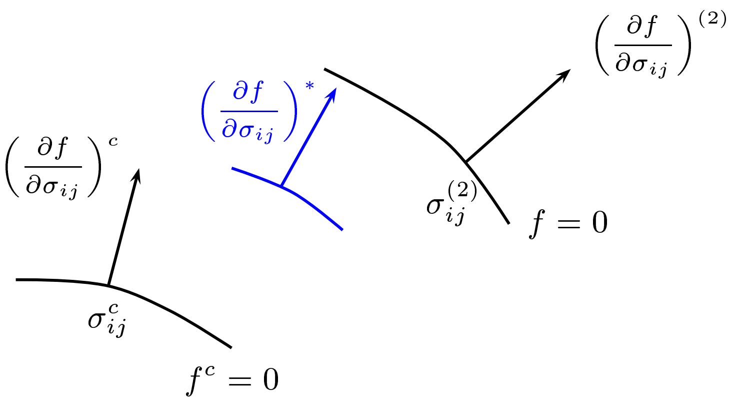
–Generalized trapezoidal rule–

Assume associated plasticity

$$\int_c^2 \frac{\partial f}{\partial \sigma_{ij}} d\lambda \approx \Delta\lambda \left(\frac{\partial f}{\partial \sigma_{ij}}\right)^*$$

$$= \Delta\lambda \left\{ (1-\theta) \left.\frac{\partial f}{\partial \sigma_{ij}}\right|_{(\sigma_{ij}^c, K^{\alpha(1)})} + \theta \left.\frac{\partial f}{\partial \sigma_{ij}}\right|_{(\sigma_{ij}^{(2)}, K^{\alpha(2)})} \right\}$$

Stress space



DIRECT CONSIDERATION TO CONSISTENCY

–Stress calculation, general situation–

Program code

- *Calculate contact stresses and strains*
- *Solve* σ_{ij}, K^α *and* $\Delta\lambda$ *from*

$$\sigma_{ij} = \sigma_{ij}^t - \Delta\lambda D_{ijkl} \left(\frac{\partial g}{\partial \sigma_{kl}}\right)^* \qquad (1)$$

$$K^\alpha = K^\alpha(\kappa^\beta + \Delta\lambda k^{\beta(*)}) \qquad (2)$$

subjected to the constraint

$$f(\sigma_{ij}, K^\alpha) = 0 \qquad (3)$$

Usual approach, from (1) and (2) derive analytically

$$\sigma_{ij} = \sigma_{ij}(\Delta\lambda)$$
$$K^\alpha = K^\alpha(\Delta\lambda)$$

Insert into the yield criteria (3), i.e.

$$f(\sigma_{ij}(\Delta\lambda), K^\alpha(\Delta\lambda)) = f(\Delta\lambda) = 0$$

non-linear equation in $\Delta\lambda$

DIRECT CONSIDERATION TO CONSISTENCY

–Isotropic hardening von Mises model–

Kinematics

$$\dot{\epsilon}_{ij} = \dot{\epsilon}^e_{ij} + \dot{\epsilon}^p_{ij}$$

Hooke's law (incr. form)

$$\dot{\sigma}_{ij} = D_{ijkl}(\dot{\epsilon}_{ij} - \dot{\epsilon}^p_{ij})$$

Isotropic material

$$D_{ijkl} = 2G\{\frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \frac{\nu}{1-2\nu}\delta_{ij}\delta_{kl}\}$$

Yield function

$$f = \sigma_{eff} - \sigma_y$$

where

$$\sigma_{eff} = \left(\tfrac{3}{2}s_{ij}s_{ij}\right)^{1/2} \qquad \sigma_y = \sigma_{yo} + K(\kappa)$$

Flow rule (associated plasticity)

$$\dot{\epsilon}^p_{ij} = \dot{\lambda}\frac{\partial f}{\partial \sigma_{ij}}$$

Evolution law (strain hardening), i.e. $\kappa = \epsilon^p_{eff}$

$$\dot{\kappa} = \dot{\epsilon}^p_{eff} = \left(\tfrac{2}{3}\dot{\epsilon}^p_{ij}\dot{\epsilon}^p_{ij}\right)^{1/2} = \dot{\lambda}$$

ISOTROPIC HARDENING VON MISES MODEL

–Integration $\theta = 1$, fully implicit–

Flow rule

$$\dot{\epsilon}^p_{ij} = \dot{\lambda}\frac{3}{2}\frac{s_{ij}}{\sigma_{eff}}$$

Approximation (integration)

$$\Delta\epsilon^p_{ij} = \int_1^2 \frac{3}{2}\frac{s_{ij}}{\sigma_{eff}}d\lambda \approx \frac{3}{2}\frac{s^{(2)}_{ij}}{\sigma^{(2)}_{eff}}\Delta\lambda \qquad (1)$$

From Hooke's law

$$\sigma^{(2)}_{ij} = \sigma^t_{ij} - D_{ijkl}\Delta\epsilon^p_{kl} \qquad (2)$$

where the trial stress is defined as

$$\sigma^t_{ij} = \sigma^{(1)}_{ij} + D_{ijkl}\Delta\epsilon_{kl}$$

Using (1) in (2) yields

$$\sigma^{(2)}_{ij} = \sigma^t_{ij} - 3G\frac{s^{(2)}_{ij}}{\sigma^{(2)}_{eff}}\Delta\lambda$$

i.e.

$$\boxed{\begin{aligned} \sigma^{(2)}_{kk} &= \sigma^t_{kk} \\ s^{(2)}_{ij} &= \frac{s^t_{ij}}{1 + 3G\frac{\Delta\lambda}{\sigma^{(2)}_{eff}}} \end{aligned}}$$

ISOTROPIC HARDENING VON MISES MODEL

–Integration $\theta = 1$, fully implicit–

We found

$$s_{ij}^{(2)} = \frac{s_{ij}^t}{1 + 3G\frac{\Delta\lambda}{\sigma_{eff}^{(2)}}} \quad \text{where} \quad \sigma_{eff}^{(2)} = \left(\frac{3}{2} s_{ij}^{(2)} s_{ij}^{(2)}\right)^{1/2}$$

Multiply each side with its self and 3/2 then take the square root

$$\sigma_{eff}^{(2)} = \left[\frac{3}{2} \frac{s_{ij}^t}{1 + 3G\frac{\Delta\lambda}{\sigma_{eff}^{(2)}}} \frac{s_{ij}^t}{1 + 3G\frac{\Delta\lambda}{\sigma_{eff}^{(2)}}}\right]^{1/2}$$

or

$$\sigma_{eff}^{(2)} = \sigma_{eff}^t - 3G\Delta\lambda \quad \text{where} \quad \sigma_{eff}^t = \left(\frac{3}{2} s_{ij}^t s_{ij}^t\right)^{1/2}$$

Internal variable $d\kappa = d\epsilon_{eff}^p = d\lambda$

$$\epsilon_{eff}^{p(2)} = \epsilon_{eff}^{p(1)} + \Delta\lambda$$

Yield criterion

Yield criterion fulfilled at state 2

i.e.

$$\sigma_{eff}^{(2)} - \sigma_y^{(2)} = 0 \quad \text{where} \quad \sigma_y^{(2)} = \sigma_y(\epsilon_{eff}^{p(2)})$$

or

$$\boxed{\sigma_{eff}^t - 3G\Delta\lambda - \sigma_y(\epsilon_{eff}^{p(1)} + \Delta\lambda) = 0}$$

ISOTROPIC HARDENING VON MISES MODEL

–Integration $\theta = 1$, fully implicit–

We found

$$s_{ij}^{(2)} = \frac{s_{ij}^t}{1 + 3G\frac{\Delta\lambda}{\sigma_{eff}^{(2)}}} \quad \text{where} \quad \sigma_{eff}^{(2)} = \left(\frac{3}{2} s_{ij}^{(2)} s_{ij}^{(2)}\right)^{1/2}$$

and

$$\sigma_{eff}^t - 3G\Delta\lambda - \sigma_y(\epsilon_{eff}^{p(1)} + \Delta\lambda) = 0$$

i.e

$$\Delta\lambda = \frac{1}{3G}(\sigma_{eff}^t - \sigma_y^{(2)})$$

Noting that $\sigma_{eff}^{(2)} = \sigma_y^{(2)}$ we find

$$s_{ij}^{(2)} = \frac{\sigma_y^{(2)}}{\sigma_{eff}^t} s_{ij}^t$$

i.e. a "scaling" of s_{ij}^t.

ISOTROPIC HARDENING VON MISES MODEL

–Integration $\theta = 1$, fully implicit, radial return–

· *Given:* $\epsilon_{ij}^{(1)}$, $\epsilon_{ij}^{p(1)}$, $\epsilon_{eff}^{p(1)}$, $\sigma_{ij}^{(1)}$, *and* $\Delta\epsilon_{ij}$

· *Calculate*

$$\sigma_{ij}^{t} = \sigma_{ij}^{(1)} + D_{ijkl}\Delta\epsilon_{kl}$$

$$\sigma_{eff}^{t} = (\frac{3}{2}s_{ij}^{t}s_{ij}^{t})^{1/2}$$

· *Determine* $\Delta\lambda$ *from* $\sigma_{eff}^{t} - 3G\Delta\lambda - \sigma_y(\epsilon_{eff}^{p(1)} + \Delta\lambda) = 0$

· *Calculate*

$$\epsilon_{eff}^{p(2)} = \epsilon_{eff}^{p(1)} + \Delta\lambda$$

$$\sigma_y^{(2)} = \sigma_y(\epsilon_{eff}^{p(2)})$$

$$\sigma_{ij}^{(2)} = s_{ij}^{(2)} + \frac{1}{3}\sigma_{kk}^{(2)}\delta_{ij} \quad \textit{where} \quad s_{ij}^{(2)} = \frac{\sigma_y^{(2)}}{\sigma_{eff}^{t}}s_{ij}^{t} \; ; \quad \sigma_{kk}^{(2)} = \sigma_{kk}^{t}$$

$$\epsilon_{ij}^{p(2)} = \epsilon_{ij}^{p(1)} + \Delta\epsilon_{ij}^{p} \qquad \textit{where} \quad \Delta\epsilon_{ij}^{p} = \frac{3}{2}\frac{\Delta\lambda}{\sigma_{eff}^{t}}s_{ij}^{t}$$