THE FINITE ELEMENT METHOD 2017
Dept. of Solid Mechanics

EXAMINATION: 2017-05-29

A maximum of 60 points can be achieved in this examination. To pass at least 30 points are required.

Permitted aid: Pocket calculator.

Anonymkod / Anonymous Code (XXX-NNNN) ....

Kurskod / Course Code (FHLXXX) ....

Personlig identifierare / Personal Identifier ....

Hand in the exam!!!!!

Problem 1: (12p)

Consider the four node element used to model an elastic boundary value problem. The nodes 2-3-4 are located along the boundary and the traction along 3-4 is sketcked below. Moreover, in point $(30, -4.5)$ a load F of magnitude $1000N/m$ is applied. The direction of the load F is $\boldsymbol{n} = 1/\sqrt{2}\,[1 \quad 1]$.

Calculate the contribution from the traction and the load F to the boundary load vector, $\boldsymbol{f}_b = \int_{\mathcal{L}_{2-3-4}} \boldsymbol{N}^T \boldsymbol{t} t d\mathcal{L}$. The thickness of the element is $5mm$.

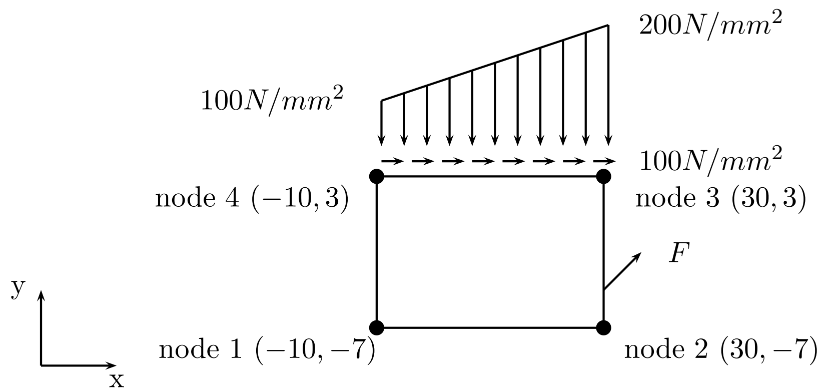


Figure 1: Four node element. All coordinates are in mm.

Problem 2: (6p)

In the element routine plani4e.m the stiffness matrix $\boldsymbol{k}^e = \int_{A_e} \boldsymbol{B}^{eT} \boldsymbol{D} \boldsymbol{B}^e t dA$ is calculated for an isoparametric plane four node element. One part of the code has been erased. State the code that has been erased !

```
  colD=size(D,2);
  if colD>3
    Dm=D([1 2 4],[1 2 4]);
  else
    Dm=D;
  end

  for i=1:ngp
      indx=[ 2*i-1; 2*i ];
      detJ=det(JT(indx,:));
      if detJ<10*eps
        disp('Jacobideterminant equal or less than zero!')
      end
      JTinv=inv(JT(indx,:));
      dNx=JTinv*dNr(indx,:);

      B(1,1:2:8-1)=dNx(1,:);
      B(2,2:2:8)  =dNx(2,:);
      B([redacted])=dNx([redacted],:);
      B([redacted])  =dNx([redacted],:);

      N2(1,1:2:8-1)=N(i,:);
      N2(2,2:2:8)  =N(i,:);

      Ke=Ke+B'*Dm*B*detJ*wp(i)*t;
      fe=fe+N2'*b*detJ*wp(i)*t;
  end

else
   error('Error ! Check first argument, ptype=1 or 2 allowed')
   return
end
%-------------------------end-------------------------------
```

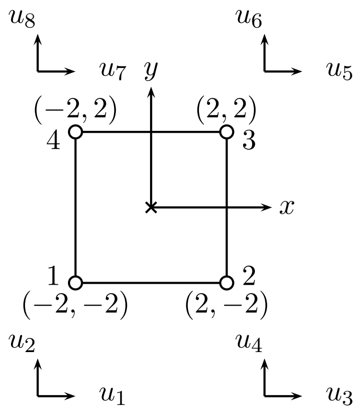
Figure 2: Part of the routine plani4e.m

Hint: The strain components in a plane analysis is given by are $\boldsymbol{\epsilon}^T = [\epsilon_{xx}\ \epsilon_{yy}\ \gamma_{xy}]$ and $\epsilon_{xx} = \frac{\partial u_x}{\partial x}$ $\epsilon_{yy} = \frac{\partial u_y}{\partial y}$, $\gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}$

Problem 3: (8p)

The element displacement vector for the element below is

$$\boldsymbol{a}^e = [0.0\ 0.2\ 0.0\ -0.2\ 0.0\ 0.2\ 0.0\ -0.2]^T$$



- Use one integration point to calculate the strain energy, $\int_A t\boldsymbol{\epsilon}^T\boldsymbol{\sigma}dA = \int_A t\boldsymbol{\epsilon}^T\boldsymbol{D}\boldsymbol{\epsilon}dA$ where $\boldsymbol{D}$ is a constant constitutive matrix. Motivate your answer.

Hint: The strain components in a plane analysis is given by are $\boldsymbol{\epsilon}^T = [\epsilon_{xx}\ \epsilon_{yy}\ \gamma_{xy}]$ and $\epsilon_{xx} = \frac{\partial u_x}{\partial x}\ \epsilon_{yy} = \frac{\partial u_y}{\partial y}$, $\gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}$.

The elasticity matrix $\boldsymbol{D}$ is given by $\boldsymbol{D} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix}$ where E and ν are constant material parameters.

Problem 4: (12p)

The governing equation for heat conduction in a one-dimensional rod subjected to convection along the constant perimeter, P, is given by

$$\frac{d}{dx}\left(Ak\frac{dT}{dx}\right) - \alpha P(T - T_\infty) = 0 \quad 0 < x < L$$

The boundary conditions are given by $q(x = 0) = \alpha PLT_\infty/A$ and $T(x = L) = T_\infty$. The 1D Fourier law reads $q = -k\frac{dT}{dx}$.

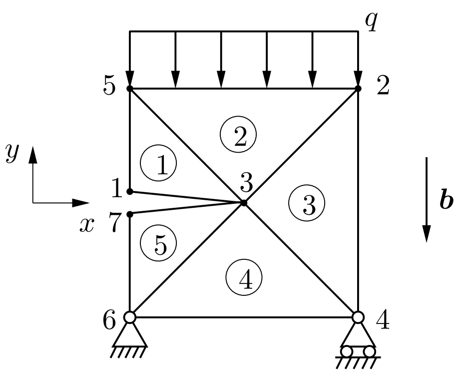
- Derive the weak form and finite element formulation for the problem.
- Use two elements of equal size to determine the temperature distribution. The convection coefficient is given by $\alpha = 16kA/(PL^2)$. All geometrical and material parameters are constant along the rod.

Problem 5: (10p)

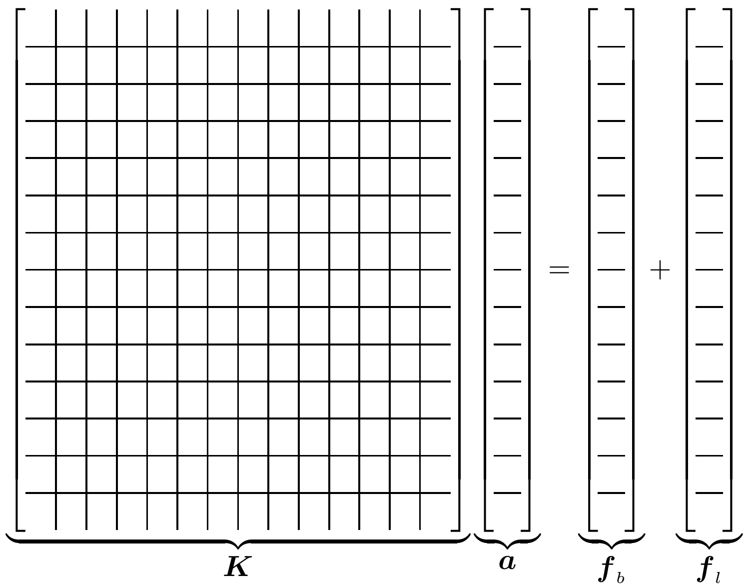
The finite element formulation for two dimensional linear elastic problem takes the form

$$\boldsymbol{K}\boldsymbol{a} = \boldsymbol{f}_b + \boldsymbol{f}_l$$

where $\boldsymbol{K}$ is the stiffness matrix and $\boldsymbol{a}$ the vector of unknown node displacements. The right hand side vectors $\boldsymbol{f}_b$ stem from the line load q and $\boldsymbol{f_l}$ from gravitation $\boldsymbol{b}$ which acts on all elements.

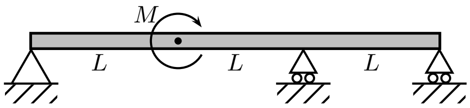


The above depicted mesh consists of five linear triangular elements. For the indicated boundary and load conditions mark in the system of equations below with
x – components that are known and different from zero
? – components that are unknown and different from zero
all blank positions are interpreted as zero.



Problem 6: (12p) (Only Pi and I (FHLF10))

A beam of length $3L$ and stiffness EI is supported and loaded according to the figure below. The moment, M, is applied at a distance L from the left support.
Determine the slope of the beam at the left support.



Problem 6: (12p) (Only F (FHLF01))
The finite element formulation for heat conduction ($div(\boldsymbol{q}) - Q = 0$) takes the form:

$$\left(\underbrace{\int_V \boldsymbol{B}^T \boldsymbol{D} \boldsymbol{B} dV}_{\boldsymbol{K}} + \underbrace{\int_{S_c} \alpha \boldsymbol{N}^T \boldsymbol{N} dS}_{\tilde{\boldsymbol{K}}} \right) \boldsymbol{a} = \underbrace{\int_{S_c} \boldsymbol{N}^T \alpha T_\infty dS}_{\boldsymbol{f}_c} \underbrace{- \int_{S_g} \boldsymbol{N}^T q_n dS - \int_{S_h} \boldsymbol{N}^T h dS}_{\boldsymbol{f}_b} + \underbrace{\int_V \boldsymbol{N}^T Q dV}_{\boldsymbol{f}_l}$$

where S_c is the boundary where convection applies, S_h is the boundary where the heat flow is prescribed and S_g is the boundary where the temperature is prescribed.

a) Show that the matrix $\boldsymbol{K} + \tilde{\boldsymbol{K}}$ is positive definite for $\alpha > 0$. The constitutive matrix $\boldsymbol{D}$ is symmetric and positive definite.

b) Show that $\mathbf{1}^T \left[- \int_S \boldsymbol{N}^T q_n dS + \int_V \boldsymbol{N}^T Q dV \right] = 0$ where V and S repesents the total volume and surface, respecively. Moreover, $\mathbf{1}$ is a vector with all entries being equal to one, i.e. the scalar product can be identified as the sum over all components.

Some hints that <u>might</u> be helpful

Hint: Green-Gauss's theorem states:

$$\int_A \phi div\boldsymbol{q} dA = \oint_{\mathcal{L}} \phi \boldsymbol{q}^T \boldsymbol{n} d\mathcal{L} - \int_A (\boldsymbol{\nabla}\phi)^T \boldsymbol{q} dA$$

Hint: Some trigonometric relations:

$$\sin(\alpha)^2 + \cos(\alpha)^2 = 1$$

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta), \quad \cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

$$\sin^2(\alpha) = \frac{1-\cos(2\alpha)}{2}, \quad \cos^2(\alpha) = \frac{1+\cos(2\alpha)}{2}$$

$$\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha), \quad \tan(2\alpha) = \frac{2\tan(\alpha)}{1-\tan^2(\alpha)}$$

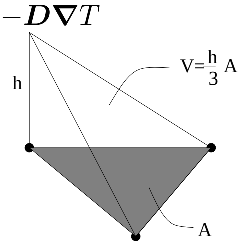
Hint: A quadratic matrix is positive semidefinite if

$$\boldsymbol{a}^T \boldsymbol{K} \boldsymbol{a} \geq 0, \quad \forall \boldsymbol{a}, \quad \text{and} \quad \boldsymbol{a}^T \boldsymbol{K} \boldsymbol{a} = 0 \quad \text{for some} \quad \boldsymbol{a} \neq \boldsymbol{0}$$

Hint: The interpolation formula of Lagrange is given by

$$l_k^{n-1} = \frac{(x-x_1).....(x-x_{k-1})(x-x_{k+1}).....(x-x_n)}{(x_k-x_1).....(x_k-x_{k-1})(x_k-x_{k+1}).....(x_k-x_n)}$$

Hint: Fourier's law is given by $\boldsymbol{q} = -\boldsymbol{D}\boldsymbol{\nabla}T$



Hint: The position, ξ_i, of the integration points and weights, H_i, for n number of integration points can be found from

n	ξ_i	H_i
1	0	2
2	$\pm 1/\sqrt{3}$	1

where integration from -1 to 1 is assumed.

Hint:

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & a & b \\ 1 & c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} (ad-bc) & 0 & 0 \\ -d+b & d & -b \\ c-a & -c & a \end{bmatrix}$$