Structural optimization mid-term 2018-02-21 solutions

Problem 1:

a) See pages 5-7 in the course literature.

b) $f'(u;\varphi) = \frac{1}{2}\int_0^1 \frac{-u'\varphi'}{\sqrt{1-\frac{1}{2}(u')^2}}dx.$

c) See page 100 in the course literature.

c) See page 192 in the course literature.

Problem 2:

a) The function is convex since $f''(x) > 0$.

b) The Hessian is $\nabla^2 f(\mathbf{x}) = 2\mathbf{A}^T\mathbf{A}$. Pre- and post multiplication with the non-zero vectors $\mathbf{y}^T$ and $\mathbf{y}$ gives $\mathbf{y}^T\nabla^2 f(\mathbf{x})\mathbf{y} = 2\left\|\mathbf{A}\mathbf{y}\right\|_2^2 > 0$. The Hessian is positive definite and therefore $f(\mathbf{x})$ is convex.

Problem 3:

The problem is equivalent to: $\mathcal{P}\begin{cases} \min\limits_{x1,x2}\ g_0 = (x_1-2)^2 + 2(x_2-1)^2 \\ g_1 = x_1 + 4x_2 - 3 \quad \leq 0 \\ g_2 = -x_1 + x_2 \qquad \leq 0 \end{cases}$

a) The Hessian of the objective function is $\nabla^2 g_0 = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$, which is positive definite. The constraints g_1 and g_2 are linear and thus convex by the definition on page 37 in the course literature. The problem is therefore convex.

b) The Lagrangian function is

$$\mathcal{L}(x_1, x_2, \lambda_1, \lambda_2) = (x_1-2)^2 + 2(x_2-1)^2 + \lambda_1(x_1 + 4x_2 - 3) + \lambda_2(-x_1 + x_2).$$

The KKT conditions are

$$\begin{aligned} \frac{\partial\mathcal{L}}{\partial x_1} &= 2(x_1-2) \ + \lambda_1 - \lambda_2 = 0, \\ \frac{\partial\mathcal{L}}{\partial x_2} &= 4(x_2-1) + 4\lambda_1 + \lambda_2 = 0, \\ \lambda_1(x_1 + 4x_2 - 3) &= 0, \\ \lambda_2(-x_1 + x_2) &= 0, \\ x_1 + 4x_2 - 3 &\leq 0, \\ -x_1 + x_2 &\leq 0, \\ \lambda_1 \geq 0, \lambda_2 &\geq 0, \end{aligned}$$

which are satisfied for the KKT point $(\mathbf{x}^*; \boldsymbol{\lambda}^*) = (x_1, x_2; \lambda_1, \lambda_2) = \frac{1}{3}(5, 1, 2, 0)$.

Problem 4:

Let the point $(x_1, x_2) = (2, -2)$ be denoted $\mathbf{x}^k$. Sensitivities and intervening variables:

$$\frac{\partial g_0}{\partial x_1}(\mathbf{x}^k) = 12 > 0 \Rightarrow y_1 = \frac{1}{U_1 - x_1},$$
$$\frac{\partial g_0}{\partial x_2}(\mathbf{x}^k) = -4 < 0 \Rightarrow y_2 = \frac{1}{x_2 - L_2}.$$

- The MMA aproximation is

$$g_0^{M,k}(x_1, x_2) = r_0^k + 12\frac{(U_1^k - 2)^2}{U_1^k - x_1} + 4\frac{(-2 - L_2^k)^2}{x_2 - L_2^k},$$

  where the constant term is $r_0^k = 12 - \left[12(U_1^k - 2) + 4(-2 - L_2^k)\right]$.

- The asymptotes and move limits should be chosen such that, for $j = 1, 2$,

$$L_j^k < \alpha_j^k \leq x_j^k \leq \beta_j^k < U_j^k.$$

- Update of asymptotes to speed up convergence: see page 68 in the course literature.

Problem 5:

$$\mathbf{B}^T = \begin{bmatrix} 1/\sqrt{2} & -1 & 0 \\ 1/\sqrt{2} & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{D} = \frac{E}{L}\begin{bmatrix} A_1/\sqrt{2} & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_3 \end{bmatrix},$$

$$\mathbf{K} = \frac{E}{L}\begin{bmatrix} \tilde{A}_1 + A_2 & \tilde{A}_1 & -A_2 \\ \tilde{A}_1 & \tilde{A}_1 + A_3 & 0 \\ -A_2 & 0 & A_2 \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} 0 & -P & P \end{bmatrix}^T,$$

where $\tilde{A}_1 = \dfrac{A_1}{2\sqrt{2}}$. The displacements are given by $\mathbf{u} = \mathbf{K}^{-1}\mathbf{F} \Rightarrow max(|u_{1x}|, |u_{1y}|, |u_{2x}|) = u_{2x} = \dfrac{2LP}{E}\Big(\dfrac{\sqrt{2}}{A_1} + \dfrac{1}{2A_2} + \dfrac{1}{A_3}\Big)$. The optimization problem is stated as

$$\mathcal{P}\begin{cases} \min\limits_{A_1, A_2, A_3} \ g_0 = \dfrac{\sqrt{2}}{A_1} + \dfrac{1}{2A_2} + \dfrac{1}{A_3} \\ g_1 = \sqrt{2}A_1 + A_2 + A_3 - V_0/L \leq 0 \\ A_i \geq 0, \quad i = 1, 2, 3 \end{cases}$$

The Hessian of g_0 is positive definite for $A_i \geq 0$, $i = 1, 2, 3$ and g_1 is a linear, thus convex function. The problem is convex. The Lagrangian function is

$$\mathcal{L}(A_1, A_2, A_3, \lambda) = \frac{\sqrt{2}}{A_1} + \frac{1}{2A_2} + \frac{1}{A_3} + \lambda(\sqrt{2}A_1 + A_2 + A_3 - V_0/L).$$

The dual problem is given by

$$\mathcal{D} \begin{cases} \max\limits_{\lambda} \ \varphi(\lambda) = 2\dfrac{(3+\sqrt{2})}{\sqrt{2}}\sqrt{\lambda} - \lambda V_0/L, \\ \lambda \geq 0, \end{cases}$$

which is solved for $\sqrt{\lambda^*} = \dfrac{(3+\sqrt{2})}{\sqrt{2}} \dfrac{L}{V_0}$. The optimal cross-sectional areas are $A_1^* = A_3^* = 1/\sqrt{\lambda^*}$ and $A_2^* = 1/\sqrt{2\lambda^*}$.