Structural optimization: Solutions to mid-term exam 190220

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Problem 1

a) See pages 148-149 in the course literature. The length between two points (x_1, y_1) and (x_2, y_2) is

$$\text{Length} = J(y) = \int_{x_1}^{x_2} \big(1 + y'(x)\big)^{1/2} dx.$$

b) See Exercise 8.2. Calculate $J'(y; \varphi) = 0$ and solve to get $y(x)$ with the conditions $y(x_1) = y_1$, $y(x_2) = y_2$, $\varphi(x_1) = \varphi(x_2) = 0$.

Problem 2

a) The Hessian is not positive (semi)definite and therefore the function is not convex.

b) The function is not twice continuously differentiable. Use the definition of a convex function on page 37 together with the triangle inequality to prove that the function is in fact convex.

c) See pages 60-61. The gradient is $\nabla f(1,2) = \big(0, -5\big)^T$, and therefore we linearize in the intervening variables $y_1 = 1/x_1$ and $y_2 = 1/x_2$. The Hessian of the CONLIN approximation of f is positive definite since the CONLIN approximation assumes that all design variables are strictly positive.

Problem 3

The problem is non-convex since the determinant of the Hessian of the objective function is negative, i.e. $\det(\boldsymbol{H}) < 0$. To solve the optimization problem, we evaluate all KKT points and investigate which of those minimize the objective function.

The Lagrangian is

$$\mathcal{L}(x_1, x_2, \lambda) = x_1^2 + x_2^2 - 3x_1x_2 + \lambda(x_1^2 + x_2^2 - 6).$$

The KKT conditions are

$$\frac{\partial \mathcal{L}}{\partial x_1} = 2x_1 - 3x_2 + \lambda 2x_1 = 0, \tag{1}$$
$$\frac{\partial \mathcal{L}}{\partial x_2} = 2x_2 - 3x_1 + \lambda 2x_2 = 0, \tag{2}$$
$$\lambda(x_1^2 + x_2^2 - 6) = 0, \tag{3}$$
$$x_1^2 + x_2^2 - 6 \leq 0, \tag{4}$$
$$\lambda \geq 0. \tag{5}$$

The limits $x_i \to \pm\infty$ are obviously not feasible since they violate the constraint in the fourth equation above.

Equations (1) and (2) give

$$x_1 = \frac{3}{2(1+\lambda)} x_2,$$
$$2x_2\big(1+\lambda - \frac{9}{4(1+\lambda)}\big) = 0.$$

This gives that either $x_2 = 0$ or $\big(1+\lambda - \frac{9}{4(1+\lambda)}\big) = 0$. Solving the last expression gives two different λ, but only one of them, $\lambda = 1/2$, satisty the KKT conditions.

The three KKT points are $(x_1, x_2; \lambda) = (0,0,0)$ and $(x_1, x_2; \lambda) = (\pm\sqrt{3}, \pm\sqrt{3}, 1/2)$. The last two points both solve the optimization problem.

Problem 4

a) From the graph we have $\frac{\partial g_0(3)}{\partial x} \approx 1.25 > 0$, i.e. g_0 should be linearized in the variable $y = \frac{1}{U-x}$ where U should satisfy $x^k = 3 < U$. For instance, choosing $U = 4$ gives the MMA approximation $g_0^{M,k}(x^k = 3) = 1.25\Big(1 + \frac{1}{4-x}\Big)$. (Indicate in the graph that $g_0 \to \infty$ as x approaches the asymptote $U = 4$).

b) From the graph we have $\frac{\partial g_0(5)}{\partial x} \approx -1.25 < 0$, i.e. g_0 should be linearized in the variable $y = \frac{1}{x}$. The CONLIN approximation is $g^{C,k}(x^k = 5) = g_0(x^k = 5) + \frac{\partial g_0(5)}{\partial x}\frac{5(x-5)}{x}$.

Problem 5

$$\begin{bmatrix} F_{2x} \\ F_{4x} \\ F_{4y} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} \iff \boldsymbol{F} = \boldsymbol{B}^T \boldsymbol{s}.$$

$\boldsymbol{D}$ is a diagonal matrix, i.e.

$$\boldsymbol{D} = \frac{E}{L}\text{diag}(A_1, A_2, A_3).$$

The stiffness matrix is also a diagonal matrix

$$\boldsymbol{K} = \boldsymbol{B}^T \boldsymbol{D} \boldsymbol{B} = \frac{E}{l}\text{diag}(A_2, A_3, A_1), \tag{6}$$

hence the inverse of $\boldsymbol{K}$ is

$$\boldsymbol{K}^{-1} = \frac{l}{E}\begin{bmatrix} \frac{1}{A_2} & 0 & 0 \\ 0 & \frac{1}{A_3} & 0 \\ 0 & 0 & \frac{1}{A_1} \end{bmatrix}.$$

The displacements are given by $\mathbf{u} = \mathbf{K}^{-1}\mathbf{F}$. The optimization problem is stated as

$$\mathcal{P}\begin{cases} \min\limits_{A_1,A_2,A_3} \; g_0 = \dfrac{1}{A_1} + \dfrac{1}{A_2} + \dfrac{1}{A_3}, \\ s.t. \begin{cases} g_1 = A_1 + A_2 + A_3 - V_0/(\rho L) \leq 0, \\ A_i \geq 0, \quad i = 1,2,3. \end{cases} \end{cases}$$

The Hessian of g_0 is positive definite for $A_i \geq 0$, $i = 1,2,3$ and g_1 is a linear, thus a convex function. The problem is convex. The Lagrangian function is

$$\mathcal{L}(A_1, A_2, A_3, \lambda) = \frac{1}{A_1} + \frac{1}{A_2} + \frac{1}{A_3} + \lambda\big(A_1 + A_2 + A_3 - V_0/(\rho L)\big).$$

The dual problem is given by

$$\mathcal{D}\begin{cases} \max\limits_{\lambda} \; \varphi(\lambda) = 6\sqrt{\lambda} - \lambda V_0/(\rho L), \\ s.t. \quad \lambda \geq 0, \end{cases}$$

which is solved for $\sqrt{\lambda^*} = \dfrac{1}{3}\dfrac{\rho L}{V_0}$. The optimal cross-sectional areas are $A_i^* = \dfrac{1}{3}\dfrac{V_0}{\rho L}$, $i = 1,2,3$.