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ROBUSTNESS ANALYSIS OF FE-MODELS

Master's Dissertation by
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Abstract

Robustness analysis of FE-models

Linda Nilsson

The Finite Element Method is used with high accuracy in many kinds of problems in the product development process today. A FE-analysis is traditionally a deterministic analysis that does not consider that the reality might differ slightly from the simulated model due to for example tolerances in the production process, uncertainties in the environment or imperfections in the material. To investigate how sensitive the design is to small changes in its geometry, material characteristics, loads etc., a robustness analysis can be added to the FE-analysis.

The thesis contains an investigation of existing methods for general robustness analyses and evaluations. Sampling methods as Plain Monte Carlo and Latin Hypercube as well as analytical reliability methods as Mean Value Method and First and Second Order Reliability methods are briefly explained. For the evaluation, regression analysis, Analysis of Variance (ANOVA) and graphical analyses are explained. Their suitability for a robustness analysis of a FE-model is discussed. The procedure proposed in this thesis for a robustness analysis is a Latin Hypercube Sampling method followed by a regression analysis, a correlation analysis and an additional graphical analysis.

A large part of the work in this Master's Thesis was the development of a tool for the evaluation phase in the robustness analysis. The tool is developed in MATLAB-environment and contains several graphical interfaces that simplifies the usage.

The proposed method is verified on three problems, one exemplary problem and two problems from the automotive industry. The first exemplary problem is a robustness analysis of an analytical function and is made to highlight the consequences of the choices made during a robustness analysis. In the second problem a cube of a foam material used in car seats is compressed with high velocity. It is investigated how different parameters influence how the stress in the cube is increased due to the compression. Material parameters and simulation settings are varied in the analysis. In the third problem the variation of the acoustic performance of a rear floor panel in a car due to small variations in the geometry of the panel is investigated. All three problem were successfully performed and single or a group of parameters with strong influence on each system response was identified.

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Chapter 1

Introduction

1.1 Background

An essential part of the development process of a new product today is virtual, experiments and tests are not made to the same extent as before. Well developed simulation techniques makes it possible to simulate a large number of variants with high accuracy to a small cost. A well performed simulation gives a result very close to the reality in many aspects. Most simulations however do not consider uncertainties caused by for example tolerances in the manufacturing process. This can be very critical since an important part of the definition of quality is that every manufactured product performs as required. It is in most engineering applications impossible to completely avoid uncertainties. Two alternative actions can be taken in avoiding unwanted variations in the product performance. To minimise the uncertainties themselves and thereby minimise the effect of them or to modify the design so that it becomes less sensitive to uncontrollable changes in the environment.

Previously the consequences of uncertainties were reduced with a security factor. The product was designed to handle loads up to ten times more than actually necessary just to be on the safe side. With the huge efforts put in today in for example the automobile industry to increase performance and reduce weight, these safety factors are not longer acceptable. It is important to find a design that handles uncertainties without a deterioration of the product performance. The goal of a robustness analysis is just that, to investigate how uncertainties affect our design, to find the parameters responsible for the variance of the result and to find ways of efficient improvement. A robustness analysis can, if made correctly and at the right time, save a lot of money later on in the development process.

Examples of common uncertainties are

- *geometrical uncertainties* such as sheet thickness due to variations in the manufacturing process
- uncertainties in the *material properties* due to manufacturing or imperfections
- uncertainties in the *boundary conditions* such as applied force, pre-tensions etc.
- *environmental uncertainties* such as temperature, wind etc.

The robustness analysis can also detect uncertainties in the FE-simulation process due to different analysis settings such as

- *increment time*
- *element controls* such as hourglass control.

1.2 Objectives

The objective with this thesis is to find a quick, simple and secure method that carries out a robustness analysis on an arbitrary simulated problem. Special focus shall lie on finding a solution that works for FE-models, especially large, complicated models and analyses where large computing times often are limiting factors.

The method shall be implemented on a couple of examples for verification and for a more profound understanding of the consequences of the choices made in the analysis.

1.3 Robustness

Many definitions of robustness and robust design exist, exactly how they are formulated depends on the specific area of application. Roughly they all say that a design is robust when the influence of uncertain factors on the performance is negligible. There are many quantities used in the definitions, commonly statistical measures are *standard deviation* and *coefficient of variance*. Other quantities are *signal-to-noise-ratios* and *probability of failure*. Some of these quantities are closer discussed in later chapters in this report.

1.3.1 Definition of robust design

It is important to find a definition of robustness that suits the area of product development. In the development of products with high demands of performance, for example in the automotive industry, a lot of resource is put on calculation and simulation of the product's quantities. The result of a calculation, the system response, can in most cases be expressed as one or more values. It is practical if the definition of robustness can be expressed with the use of these values.

Two things should be fulfilled for a robust design.

- The mean value of the system response shall not deviate too much from the desired value of the system response.
- The variation of the system response, due to uncertainties in the environment, must be small.

Exactly what is meant by "not too much" and "small" must be defined individually for each model and each system response. Higher demands are for example set on the breaks in a car than on a vacuum cleaner. Higher demands are often also set on stresses than on mass. Statistical quantities useful in the definition of variation are for example *standard deviation* and *coefficient of variance*. See section 2.2 for a discussion of the suitability of these quantities.

1.3.2 Robustness analyses

To define the robustness of a design, an analysis must be made. Three different types of analyses can be distinguished; *sensitivity analysis*, *reliability analysis* and *robustness analysis*. The objective of the *sensitivity analysis* is to find factors with strong influence on the results. A sensitivity analysis shows where it is meaningful to put extra effort in reducing the variance of the input variables. The result of a *reliability analysis* is the probability that the output of interest exceeds a certain limit. This is an important measurement of quality, for example the well-known “Six Sigma”-criterion is based on a reliability analysis. A *robustness analysis* combines these two analyses by considering the probability functions of the uncertain factors in the sensitivity analysis. It is possible to see how the output varies with uncertainties in the environment and to identify the factors responsible for this variation. The robustness analysis is normally more time-consuming than the others but the result is very useful in a quality improvement.

For a successful analysis, independent of which one, the work is divided into following parts:

- Define the objective of the analysis and the critical measurements that defines whether the design is robust or not.
- Choose the parameters to be considered as uncertain and choose a distribution function for each of them.
- Choose an analysis method.
- Collect data through experiments or simulations.
- Analyse the collected data and draw conclusions.

The first two parts of the analysis are dependent on the specific problem. A poorly defined goal or insufficiently chosen parameters can lead to results that have less or none connection to the reality. A thorough investigation and discussion is recommended before the uncertain parameters and their distributions are set. This part is especially important in reliability and robustness analysis since the distributions of the input normally have a big influence on the distribution of the output. The choice of variables and distributions are discussed in section 4.1 and the consequences can be seen in the example in section 6.1.

The next step, choosing a method, demands some knowledge of the problem. There are several methods available and they all have advantages for different forms of analysis models and goals. Some are more suitable for analysis of experimental data others for analysis of simulations, some for a sensitivity analysis others for a reliability analysis. A more closely description of the most common methods is presented in section 2.1.

For the actual collection of data, many techniques are used. The choice of technique mainly depends on the model to be analysed and the system response but also of the

method chosen for the robustness analysis.

Finally the gathered data must be evaluated in a suitable way. This is an important part of the analysis and depends strongly on the skill of the engineer. Different statistical methods can be used and a basic understanding of those is essential. A brief discussion of the available methods and measurements is found in section 2.2.

Chapter 2

Theory

2.1 Methods for Robustness and Reliability analysis

2.1.1 Sampling Methods

To randomly collect a number of samples and let the population they build represent a continuous interval is called a sampling method.

In the computer simulation this means that the calculation is repeated an arbitrary number of times. The values of the uncertain variables in each computation are chosen randomly within their defined distributions. Sampling is a simple and well known method for parameter studies and robustness evaluations. One advantage of the sampling methods is the ability to perform the simulations and experiments parallel. Another is that they are model independent i.e. the sampling methods can be used on any problem and goal. The disadvantage of sampling methods is the, due to the randomness in the sample generation, difficulty to guarantee the representation of the entire variable space. This uncertainty can be reduced by choosing a large sample size or by the use of special algorithms.

Plain Monte Carlo sampling

The *Plain Monte Carlo Sampling method* PMC is the most simple way to generate the samples. The values of the input data in the samples are completely randomly chosen [GIUNTA *et al.* 2003]. A large amount of combinations is needed to guarantee a reliable result. 10 samples generated with PMC are displayed in figure 2.1. The main problems with PMC are the risk of pattern formation on the population, so called *clusters*, and the difficulties to represent the entire variable space. To minimise these risks, and by that reduce the demanded sample size, different alternatives to the PMC have been developed. One of them is Latin Hypercube sampling (LHS).

With a simple probability factor on the random choice, the plain Monte Carlo sampling method can be used with uniformly as well as with non-uniformly distributed variables.

Latin Hypercube sampling

By dividing the interval in a number of bins and guarantee that each row and each column of bins gets a sample, the entire interval gets represented with a much smaller number of samples than with PMC [GIUNTA *et al.* 2003]. This is called *Latin Hypercube Sampling* LHS. It is randomly chosen which bin in a column/row gets a sample. The exact location of each sample in the bin is also randomly chosen. 10 samples generated with Latin Hypercube Sampling (LHS) is displayed in figure 2.2. An algorithm that reduces unwanted correlation between the input variables is there implemented.

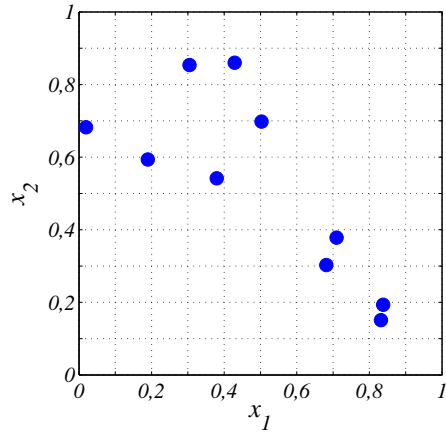


Figure 2.1: Example of 10 samples created with Plain Monte Carlo. The two variables are defined with a uniform distribution in the interval $[0,1]$. Note that large areas of the interval completely lack samples.

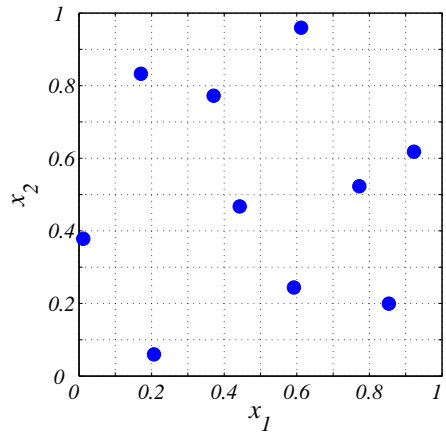


Figure 2.2: Example of 10 samples created with LHS. The two variables are defined to be uniformly distributed in the interval $[0,1]$. Note how the population fulfil the criterion that each row and each column contains one sample. The entire interval is well represented with samples.

It can easily be seen in figure 2.3 that the distribution of the samples with Latin hypercube sampling is superior to plain Monte Carlo with the same number of samples.

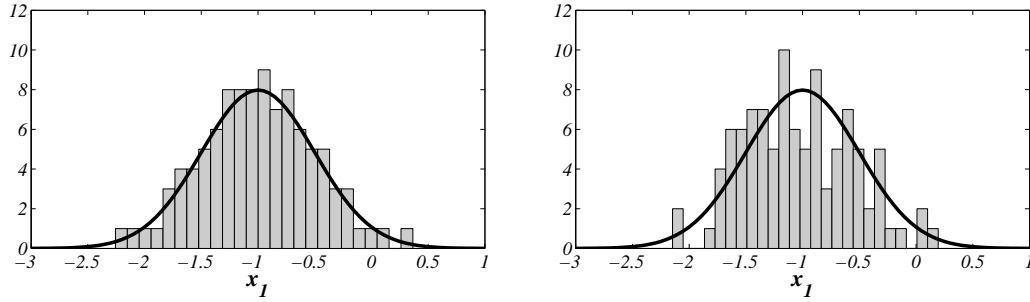


Figure 2.3: Histogram of 100 samples created with LHS and PMC respectively, $\bar{x} = -1$ $\sigma_x = 0.5$. The curve is the distribution function for the normal distribution $f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\bar{x})^2/2\sigma^2}$.

2.1.2 Analytical Reliability Methods

In a reliability analysis the probability of failure of a system, i.e. the probability that one or more system responses exceed a certain limit, is calculated. This limit of the response is described in a *limit state function* $f(x_1, x_2, \dots) \leq z$. A simple example is the probability of plastic strain in a structure, the limit state function is then defined as *stress* \leq *yield stress* or $\sigma(F, E, x_1, \dots) \leq R_e$.

In most cases, the probability of failure is very small. To guarantee a certain level of accuracy of the probability of failure with a sampling method (a certain amount of samples in the failure area) a very large amount of samples is needed. To make the analysis more efficient a number of alternatives to the original Plain Monte Carlo sampling method have been developed. Some methods are based on the PMC, such as *importance sampling* [OLSSON 2002] or *directional sampling* [BUCHER 2005B]. Other methods approximates mean and variance of the response with different analytical algorithms. They are called *analytical reliability methods* [ELDRED *et al.* 2001].

Mean value method

The *Mean value method* is the most simple of the analytical reliability methods. It analyses the response in one point, where all the variables lie in their means respectively. With the response value and the gradients of the response in this point, a probability function is approximated. This is, of course, a very rough approximation but can, with extreme small calculation costs, give relatively accurate results if the response functions are nearly linear.

First and Second Order Reliability Method

First and Second Order Reliability Methods (FORM/SORM) are more complex than the Mean value method. The uncertain variables are transformed so that they are uncorrelated to each other and have normal distributions. The transformation is made to simplify the search for the *most probable point of failure, MPP*. The search is often made with an optimisation process with the goal of minimising the distance β from the sample mean to the area of failure. The probability of failure is calculated through integration of the distribution function outside the limit state function. In FORM, the exact limit state function is approximated to a linear function and in SORM to a polynomial of second order. The approximated limit state functions contains *MPP* and is often approximated with the function gradients in *MPP*. The biggest advantage of FORM and SORM in comparison to the sampling methods is that the computing cost is independent on the probability of failure. [ELDRED *et al.* 2004]

An example where FORM is implemented is presented. The probability that the function $f(x_1, x_2) = x_1 + x_2$ does not exceed 1 is the required result. The input variables x_1 and x_2 are normally distributed with $\bar{x} = 0$ and $\sigma = 1/3$. Furthermore x_1 and x_2 are statistically independent which makes a transformation unnecessary. The limit state function is $f(x_1, x_2) \leq 1$. In figure 2.5 the probability density function for two normally distributed functions is displayed. The limit state function is marked red in the figure. With an optimisation process, the most probable point of failure *MPP* is located. In this point a linear (in FORM) approximation of the limit state function is placed. In this example the limit state function is already linear and if the *MPP* is successfully found the approximation of the probability of “failure” will be an exact value. The probability of $f(x_1, x_2) \leq 1$ is approximated as the volume of the probability density function up to the approximated limit state function and calculated through integration.

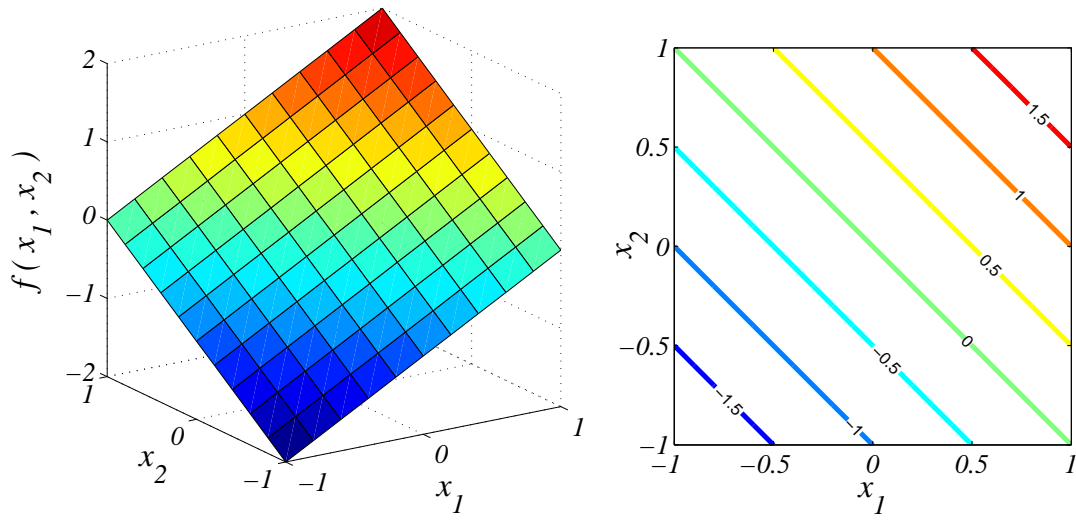


Figure 2.4: The function $f(x_1, x_2) = x_1 + x_2$ displayed as surface and contour plot.

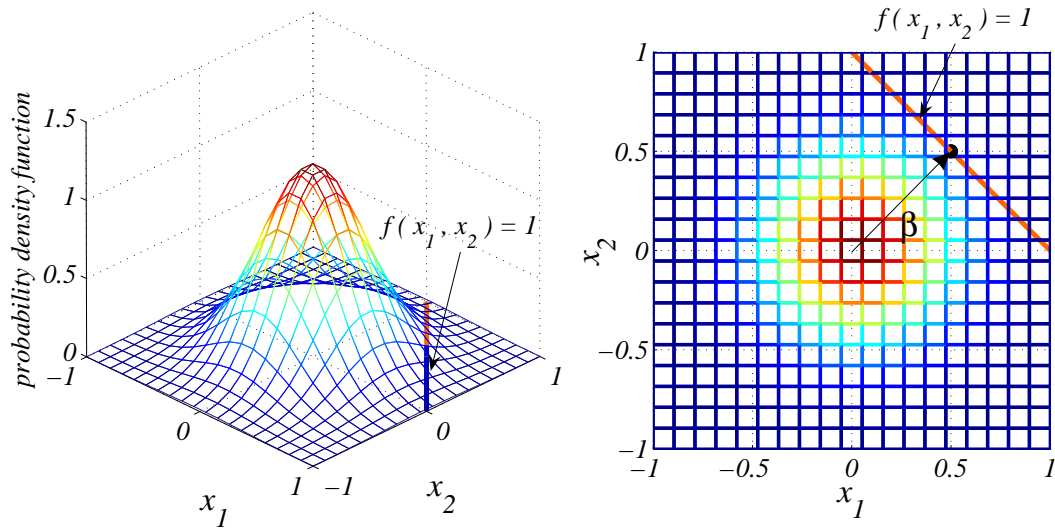


Figure 2.5: The 2D-probability density function of the two variables x_1 and x_2 . The limit state function $f(x_1, x_2) = 1$ is marked in both plots.

2.2 Statistical methods for data evaluation

The analysis of the received data is just as important as the analysis method. In the case of robustness evaluations of FE-models the collection of data can be expensive. It is important that we find a method to evaluate the data in an accurate and secure way. The methods will be briefly discussed in this chapter for a better understanding of the choices made in the next chapters. For a more profound explanation see for example [BETHEA *et al.* 1995].

2.2.1 Regression Analysis

A regression analysis uses experimental or through simulations computed data to approximate a function that shows a possible dependence between two or more variables. The characteristics of the function is beforehand defined by the user. The most simple approximation is a linear function, but if necessary, a polynomial or an exponential function can be chosen.

A linear approximation of the dependence between two variables is called a *simple linear regression model*. The relationship is written as

$$Y = \beta_0 + \beta_1 X + \varepsilon \quad (2.1)$$

where the *regression coefficients* β_i are chosen so that the *random error* ε is minimised. For a more precise description of how this is performed see [BETHEA *et al.* 1995].

Analogue to the simple linear model, a multiple linear model is based on

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \dots + \beta_p X_p + \varepsilon \quad (2.2)$$

and a polynomial model is based on

$$Y = \beta_0 + \beta_1 X + \beta_2 X^2 + \dots + \beta_p X^p + \varepsilon. \quad (2.3)$$

The regression analysis is followed by a correlation analysis where the *coefficients of correlation* are determined. The correlation coefficients are a measurement of how well the curve fits to the samples i.e. how well the approximated regression model explains the relationship between the variables. The correlation coefficient is always a value between -1 and 1, where 1 is a positive slope and -1 is a negative slope. The higher the magnitude, the clearer relationship. A closer description of the correlation coefficients with additional examples is found in section 2.2.4.

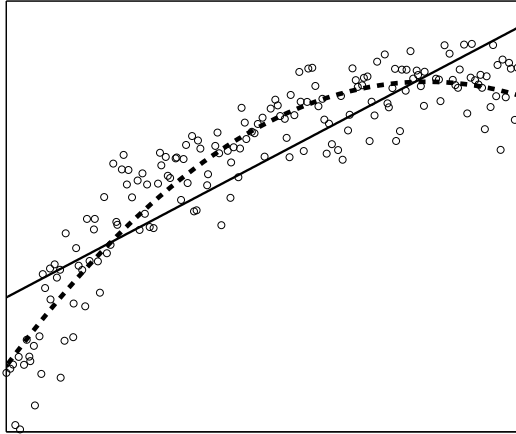


Figure 2.6: Regression analysis of 176 samples, linear approximation (dotted) with correlation coefficient $r = 0,87$ and polynomial approximation of degree 2 (dashed) with $r = 0,94$.

2.2.2 Analysis of Variance, ANOVA

An alternative to the regression analysis is the *analysis of variance*. Instead of the regression model in equation 2.1 is the response Y_{ij} in ANOVA for three dimensions modelled as

$$Y_{ij} = \mu + \alpha_i + \beta_j + \varepsilon_{ij} \quad (2.4)$$

where μ is the contribution to the response from the mean, α from one of the independent variables, β from the other variable and ε is the experimental error (in the case of computer simulations is ε zero).

The investigated variables have an influence on the response if α and/or β are significantly different from zero. Analogously, the test can also be performed when more than two independent variables are investigated, and with small alterations, interactions between variables can be detected. The calculations used to determine μ , α , β and ε are rather complex and will not be explained here, the reader is referred to [BETHEA *et al.* 1995] for a more profound description.

The samples used in ANOVA have been generated in a certain pattern. The variables are discretised and for an optimal result all the possible combinations of the input variables shall be tested. It is possible to reduce the number of combinations for example by using a fractional factorial experiment with a reduction of cost as a result. If the analysis is properly modelled the analysis cost can be reduced with an acceptable decrease in accuracy.

ANOVA can also be used as an alternative to a correlation analysis to estimate the “goodness of fit” of the curve obtained in a regression analysis. This use of ANOVA will not be investigated within this thesis.

2.2.3 Scatter plot analysis

Two or three dimensional *scatter* or *ant-hill plots* of the samples are often very informative when it comes to identifying relationships and interactions between variables.

The following figures show how to interpret a scatter plot.

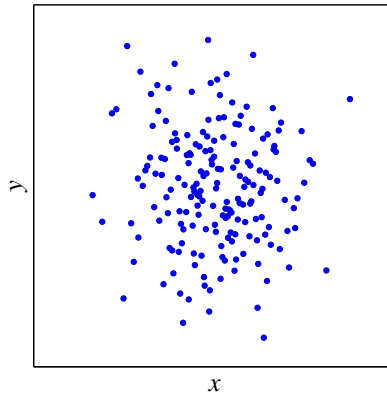


Figure 2.7: This scatter plot indicates that x has no influence on y . The linear coefficient of correlation of x with respect to y is $r = -0.06$.

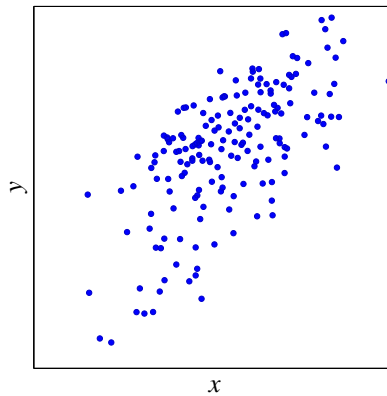


Figure 2.8: This scatter plot shows a linear relationship between x where also other variables have influence on y . The linear coefficient of correlation of x with respect to y is $r = 0.66$.

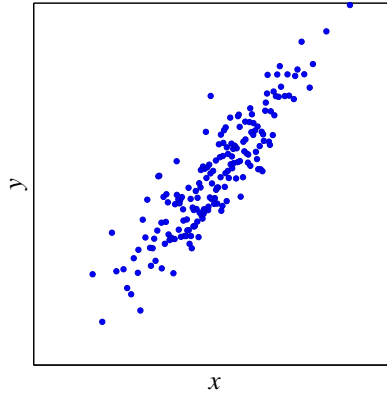


Figure 2.9: This scatter plot shows a strong linear relationship between x and y . Other variables have very little influence on y . The linear coefficient of correlation of x with respect to y is $r = 0.90$.

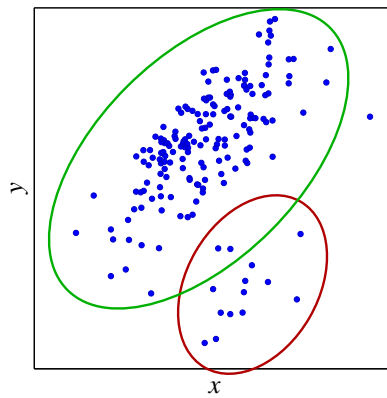


Figure 2.10: Two clusters can be identified in this scatter plot. The two variables are linear related in the large cluster, the samples in the other cluster are affected by another variable or interactions between variables.

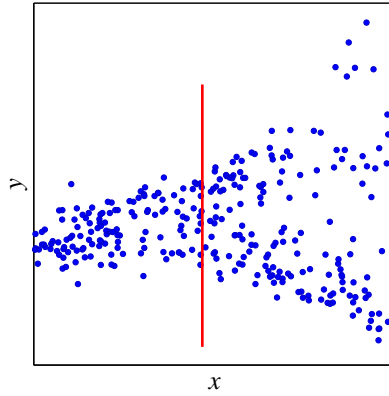


Figure 2.11: This scatter plot shows an example of a response y that is robust when x is small but strongly influenced by one or more other input variables when x is large.

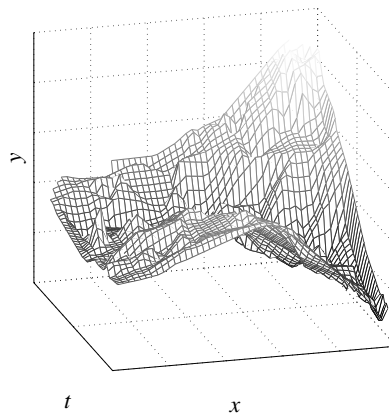


Figure 2.12: An approximated response surface of the same sample population as in figure 2.11. The input variable t have strong influence on y when x is large, but small influence when x is small.

2.2.4 Statistical measurements

For a better understanding of the statistical quantities used and the choices made in this report, a brief description follows.

Mean value

The *mean value* \bar{x} is a measurement of where the samples are located. It is a very simple measurement, but still one of the most important. The mean is defined as the sum of the samples divided by the number of samples

$$\bar{x} = \frac{\sum_{i=1}^n x_i}{n} \quad (2.5)$$

Variance

For a description of how the data is spread around its mean, the variance is used. The *variance* σ_x^2 is the sum of squares of the deviations of the data from the mean, divided by the number of samples - 1.

$$\sigma_x^2 = \frac{\sum_{i=1}^n (x_i - \bar{x})^2}{n - 1} \quad (2.6)$$

The *standard deviation*, σ_x is also commonly used in this context and is defined as the square root of the variance. A variable with normal distribution has 68% of its samples in the interval $\bar{x} \pm \sigma_x$.

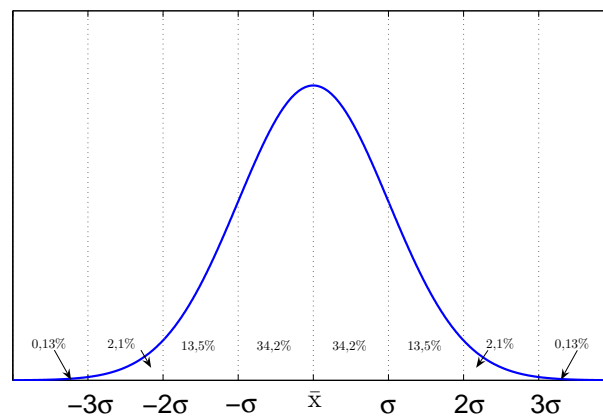


Figure 2.13: Example of a normal distribution with mean and standard deviation

Another measurement is the *coefficient of variance*, CV, defined as the standard deviation, divided by the mean value. The coefficient of variance can be useful when comparing the variance of two or more variables with different means. Notice that with a sample mean close to zero the coefficient of variance can be misleading.

Coefficient of Correlation

The *Pearson's correlation coefficient* r is a measurement of how much the variation of one variable depend on the variation of another variable. The correlation coefficient can be received after a regression analysis where a dependent variable is approximated to be a function of a number of independent variables.

Mathematically, the correlation coefficient is a measurement of how well the curve derived in a regression analysis fits to the samples, i.e. how far off from the curve the samples are located. A scatter of the samples around the approximated curve depends on one of two things. Either the curve is badly approximated or the response is affected by other variables. Other influent variables "draw" the samples from the curve and the correlation coefficient decreases. See section 6.1 for an example. The linear correlation coefficient between two variables is the most simple coefficient and is defined as

$$r_{xy} = \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^n (x_i - \bar{x})^2 \sum_{i=1}^n (y_i - \bar{y})^2}} \quad (2.7)$$

For a definition of non-linear correlation see [BETHEA *et al.* 1995].

The calculated correlation coefficient is only an estimation and a rather uncertain measurement. Due to the randomness in the sample generation it is possible to receive distinctly different estimated correlation coefficients in two different populations with the same distributions and dependencies. It is important to be aware of this uncertainty during the evaluation. The measure to express the uncertainty is called the *confidence interval* of the correlation coefficient and is depending on the magnitude of the correlation coefficient and the number of samples. A high correlation coefficient has a smaller confidence interval i.e. is more reliable than a low correlation coefficient.

The distribution of the correlation coefficient is not entirely normal, due to the limits at -1 and 1, but slightly skewed toward zero. The confidence interval of a correlation coefficient of a totally random variable (samples chosen with PMC) can be calculated with the statistics of Fisher's z -transformation. It transforms the distribution of the correlation coefficient to a normal distribution with mean $\bar{z} = 0.5 \cdot \ln((1+r)/(1-r))$ and variance $\sigma_z^2 = 1/(n-3)$ where n is the number of samples [BETHEA *et al.* 1995]. The 95%-confidence interval of for example an estimated correlation coefficient $r = 0.5$ of a population with 300 samples generated with Plain Monte Carlo is 0.170 [BUCHER 2005A]. That means that we are 95% confident that the actual correlation coefficient lies within

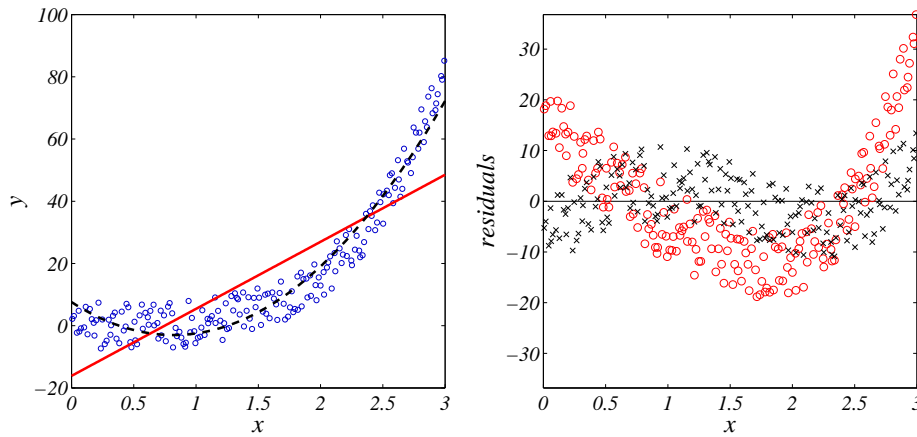


Figure 2.14: The pattern in the size of the residuals after the linear regression analysis (o) indicates a badly approximated regression curve. After a quadratic regression analysis (x) the approximated curve is better approximated and the residuals are evenly distributed.

in interval of 0.170 around 0.5. A correlation coefficient is called *significant* if the confidence interval of the correlation coefficient does not contain $r = 0$ i.e. if, with 95% certainty, an actual relationship does exist.

The correlation coefficient, r , will always be a value between -1 and 1 where a negative value means a negative slope and a positive value a positive slope. A correlation coefficient close to 1 or -1 respectively means a strong relationship. A low magnitude of the correlation coefficient either means that no relationship exists or that the regression curve is badly approximated. Three analyses help the user to determine the cause of a low correlation coefficient, a residual analysis, the sum of coefficients of determination or a visual analysis of scatter plots.

The *residual analysis* show how the residuals, the distances between the sample and the approximated curve, depend on the independent variable. A pattern indicates a badly approximated regression curve. Randomly large residuals mean that the regression curve is well approximated but the influence of other variables causes a spread of the samples, see figure 2.14.

The *coefficient of determination* r^2 is the squared correlation coefficients. The coefficient of determination is a measurement of how big part of the output variance can be explained by the variance of the investigated input. If the *sum of the coefficients of determination* R^2 for all input variables with significant correlation coefficients with respect to one response function is close to 100%, then the chosen regression analysis explains all the variation in the model. The received correlation coefficients can be trusted. If not, a relationship can exist even though the correlation coefficients are low. Another regression model must be tested. Alternatively, the variation is caused

by one or more uncertain variables that are not included in the analysis. This is more common in physical experiments than in computer simulations since most parameters in the latter case are constants if not defined otherwise. It should also be noted that it is impossible to get a sum above 100% if the correlations between the input variables truly are zero, i.e. two independent variables can not have a correlation of 0,8 with the same dependent response. Corrections have to be made if some of the input parameters are correlated to each other. It is for example after a quadratic correlation coefficient. This is for example often the case after a nonlinear regression analysis since x_i^2 and x_i are correlated to each other if $\bar{x}_i \neq 0$.

One possibility to calculate the *sum of coefficient of determination* for correlated input variables is to perform a *Principal Component Analysis, PCA* [JOLLIFFE 1986]. The PCA creates new variables that are linear combinations of the old variables and linearly independent of each other. With these new variables a regression analysis with a following correlation analysis is made and the *sum of coefficient of determination* can be calculated as the sum of the squared correlation coefficients as usual. This will not be made in this thesis, the *sum of coefficient of determination* will not be used when the input variables are correlated to each other.

Partial correlation

A complement to the correlation coefficient is the *partial correlation coefficient* $r_{x_i y | x_j}$. As mentioned earlier, the sum of the coefficients of determination never exceeds 100%. When a large number of variables have equally large influence on the output, the Pearson's correlation coefficients gets very small and can be a poor measurement of relationship. A better choice can be the partial correlation. It measures the relationship between two variables when the influence of other variables is removed. For a partial correlation between x_1 and y where only one other variable x_2 is held fixed, the definition of the partial correlation coefficient is

$$r_{x_1 y | x_2} = \frac{r_{x_1 y} - r_{x_1 x_2} r_{x_2 y}}{\sqrt{(1 - r_{x_1 x_2}^2)(1 - r_{x_2 y}^2)}} \quad (2.8)$$

where r_{ij} is the Pearson's correlation coefficient for variables i and j . The input variables are often independent of each other, i.e. $r_{x_1 x_2} = 0$, and the partial correlation is

$$r_{x_1 y | x_2} = \frac{r_{x_1 y}}{\sqrt{(1 - r_{x_2 y}^2)}} \quad (2.9)$$

When a partial correlation with more than one other variable held fixed, a *multiple regression analysis* is needed. The partial correlation coefficient is then calculated with help from t -statistics and residuals. See [BETHEA *et al.* 1995] for a closer explanation.

Coefficient of Rankcorrelation

A second complement or alternative to the Pearson's correlation coefficient is the *Spearman's rankcorrelation coefficient* r' . The rankcorrelation shows if the output is monotonically increasing or decreasing with an increased input variable. If the output gets bigger for every increase of the input, then the coefficient of rankcorrelation is 1. If the output gets smaller by increasing the input then the coefficient of rankcorrelation is -1. The definition is

$$r' = 1 - 6 \sum_{i=1}^n \frac{d_i^2}{n(n^2 - 1)} \quad (2.10)$$

where d_i is the difference in rank between y_i and x_i and n is the total number of samples. Note that the rankcorrelation is not affected by the slope of the curve, i.e. it does not take the variance of the output in consideration.

The rankcorrelation is independent on the way a monotonic function is increasing or decreasing, i.e. both a linear increase and an exponential increase can have a rankcorrelation coefficient of 1.

Chapter 3

Robustness analyses of FE-models

3.1 Advantages and possibilities

Performing robustness analyses on FE-models instead of on prototypes creates many possibilities but also a few special demands and limitations. When a robustness analysis of a finite element model is to be executed it is important to remember that the model and calculation themselves are uncertain factors. The finite element method is a numeric method and the result can for many reasons be quite far from the reality. It can be meaningful or perhaps even necessary to perform a robustness analysis of the calculation itself before analysing the effect of geometrical and environmental uncertainties. An example of an robustness analysis of the simulation itself as well as of the design of the product is described in [WILL *et al.* 2005].

The advantages of a robustness analysis are obvious. In an early stage of the development process it is possible to investigate if the product will fulfil the demands of quality. The well-developed simulation techniques used in the development today has made it possible to optimise each small part with respect to stiffness, strength and weight. One problem with optimisation is that it often leads to instability and that the finished product does not show the same qualities as the simulated model. A robustness analysis gives a relatively quick and secure answer to the question if the model can be implemented in the real world or not i.e. if it is sensitive to variations we can not foresee. A design that shows good qualities in the simulation but does not keep the same standard once being produced is worthless and all the work in the optimisation was useless.

3.2 Demands and limitations

Rather few special demands for robustness analyses of FE-models are found. Two special cases that are slightly more difficult are large FE-models or advanced analyses with long computing time and models with several uncertain parameters.

3.2.1 Robustness analyses with large FE-models

Large FE-models mean long computing time and time is a limited factor in the product development process today. A robustness analysis on large FE-models is especially critical due to the multiple repetition of the calculation in the analysis. This puts high demands on the analysis method. It must produce enough information about the response behaviour to estimate the statistics with high precision in short time. It also puts demands on the evaluation. The methods used to reduce the total computing time often leads to a reduction of the information available in the evaluation. The evaluation method must be able to produce a reliable result with this reduced information.

3.2.2 Robustness analyses with several uncertain parameters

A large number of uncertain parameters does not necessarily mean large computation time for each simulation and it puts other demands on the robustness analysis than large models. The most demands are set on the evaluation of the collected data. With few variables it is easy to use several evaluation techniques as for example regression analysis and plots in the search for relationships. An evaluation of the data in an analysis containing hundreds of variables and system responses can be more difficult. It is for example more difficult to interpret the correlation coefficients when several variables have influence on the system response. How to choose which variables to be included in the analysis is discussed in chapter 4.1.

Chapter 4

Proposed procedure for a robustness analysis of a FE-model

4.1 Choice of variables and their distributions

The choice of variables and their distributions in the analysis is very important for the usefulness of the result and can sometimes be very difficult. Two decisions have to be taken.

- Which variables shall be set as uncertain and included in the analysis?
- How is the uncertainty for these variables best described?

The choices are depending on the specific problem but some general aspects must be taken under consideration before a decision can be made.

4.1.1 Choice of variables

When performing a robustness analysis with experimental data, all uncertain parameters are automatically varied. With simulated data, on the other hand, only the parameters defined as uncertain are varied. If a variable that is uncertain in reality is considered constant in the simulation, then the result from the analysis represents the reality poorly. Unless the simulations are compared with experiments it is impossible to guarantee that the choice of included variables is sufficient. From this point of view, as many variables as possible should be included in the analysis.

On the other hand is an evaluation of the data often more complicated with many uncertain variables. As mentioned in section 2.2.4 the correlation coefficients are more difficult to interpret when many variables influence the system response. The solution can be to perform several analyses where only a few parameters are varied in each analysis. The variance of the response in each analysis is then evaluated together with the existing correlations. This is of course extremely time consuming and a risk that important interactions between the variables are lost is present. Although, a division into subproblems can be the only way to determine the influence of each separate variable when the model contains many uncertainties.

4.1.2 Choice of distribution

How important the choice of distribution is depends on the main objective for the analysis. If the main objective is to identify the influence of the different parameters, then the distribution are less important. If the main objective is to determine the reliability, then the distributions are of very large importance.

Most uncertainties are best described with a normal distribution, but sometimes a uniform or a log-normal distribution is a better choice. Normally distributed variables have a 68%-probability that its value lies in the interval $\bar{x} \pm \sigma$, a 95%-probability within $\bar{x} \pm 2 \cdot \sigma$ and a 99%-probability within $\bar{x} \pm 3 \cdot \sigma$, see figure 2.13. The probability for

a variable value below 0 with a log-normal distribution is always zero, which often is the case of uncertain variables in engineering problems. The choice of distribution is depending on the variable and measurements are often a good base for the decision. For example if a sheet thickness is considered an uncertain variable, then it is relatively easy to measure the thicknesses of a number of sheets and then fit a distribution function to the measured data.

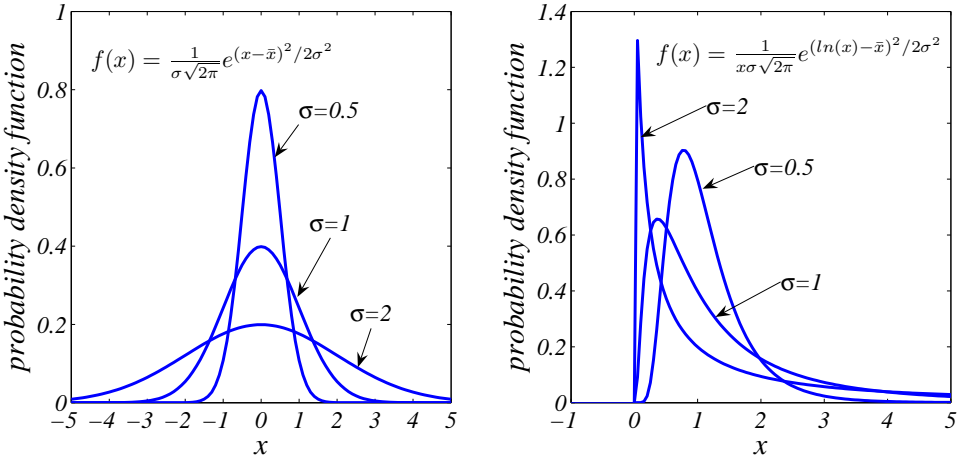


Figure 4.1: Probability density functions of normal (left) and log-normal (right) distributions with $\bar{x} = 0$ and $\sigma = 0.5, 1$ and 2 respectively.

4.2 Recommended selection of analysis method

In section 2.1 different methods for robustness and reliability analyses are described. The following section will discuss how well these methods fulfil the demands defined in chapter 3.2 and how they are best used for our purposes. The methods discussed are *sampling methods* and *analytical reliability methods*.

4.2.1 Suitability of the available methods

Stochastic sampling methods

The *stochastic sampling methods* are rather time consuming but have many important positive qualities. The possibility to run simulations parallel reduces the disadvantage of the long calculation time. The model independence and easy implementation on already existing models and softwares makes a stochastic sampling method the best choice for fulfilling the demands on a robustness analysis of a FE-model. Once the samples are created, they can be used for a large number of analyses and a lot of different conclusions can be drawn. The influence of the number of samples, the *sample size*, on the reliability of the statistical quantities is tested in section 4.2.3.

As for the choice of which sampling method to use, all tests show that Latin Hypercube is superior to Monte Carlo. A small disadvantage with LHS is the slightly more complicated way to choose the input parameters for the samples. With a suitable software, this disadvantage is negligible. What should be noticed is the danger in analysing only a part of the data collected with Latin Hypercube. The number of bins that guarantees the representation of the entire interval is defined before the sample generation. To stop the calculations before all simulations are made can lead to that the interval no longer is fully represented with samples. It should also be noticed that to start with 100 samples and then run an additional 50 samples with LHS is not the same as running 150 samples at once. It can be possible to add a convergence test or a significance test to the Monte Carlo analysis that reports when the confidence intervals for the requested statistical quantities are small enough. With a test of convergence or significance unnecessary simulations can be avoided and the computing time can be decreased.

Analytical reliability methods

The *analytical reliability methods* can be very useful when the only requested result is the probability that a certain event occurs. In these cases the computing time needed is much smaller than with a sampling method. Two demands are set on the problem for a successful analysis with the analytical reliability methods. It is necessary to express the response limit in a limit state function of the form $f(x_1, x_2, \dots) \leq z$ where z is

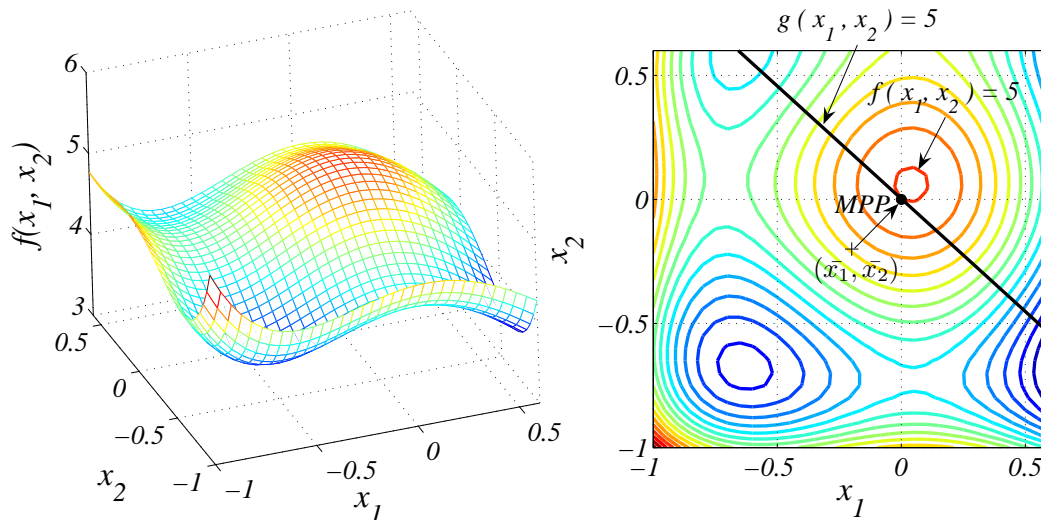


Figure 4.2: An example of a problem where an analytical reliability method is badly suited. The approximated area of failure, outside of $g(x_1, x_2) = 5$ is much larger than the actual area of failure inside of $f(x_1, x_2) = 5$.

a constant. FORM and SORM, see section 2.1.2, search for the most probable point of failure MPP and approximate the limit state function in this point. All points “outside” of this function are considered failures. This means that the analytical reliability methods do not work when the response contains single peaks where the limit is exceeded. The response must be monotonic to guarantee a correct result, see figure 4.2. The second demand is set by the optimisation in the search of the MPP. Most optimisation processes use gradients so it is necessary that these are obtainable. The mean value method also requires gradients.

The probability of failure is the only result from the analytical reliability methods. It is difficult to derive the origin of the variation that causes this probability. Most of the time the main objective is to identify the variables that affect the response the most and therefore it is recommended to perform the robustness analysis with a sampling method. It might be practical to start with a reliability analysis to approximate the variance of the response to see if a robustness analysis is needed. The influence of the variables is only important when the variance is too large and a reduction is required.

4.2.2 Software

A robustness analysis is meaningful on a large number of different models that must be simulated with different softwares. It is therefore important that the method is kept as general as possible. For this purpose the open-source software DAKOTA [3] is very well suited. DAKOTA generates the samples and control the iteration loops with

the simulation code. The only demand DAKOTA puts on the software that performs the actual calculation/simulation is the ability to perform the parametrisation through and present the result in ASCII-files. As a result DAKOTA produces an ASCII-file containing the values of the input variables and the calculated responses that later can be evaluated. All the previously mentioned methods for robustness analyses are available in DAKOTA.

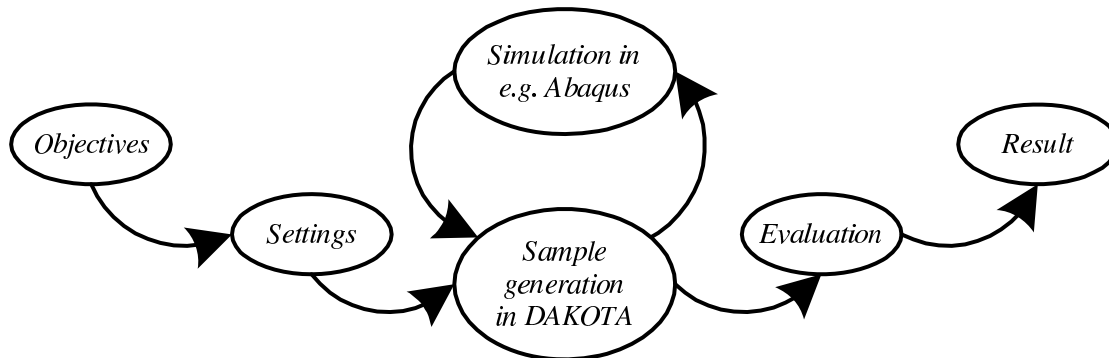


Figure 4.3: The process of a robustness analysis with DAKOTA and the simulation software in an iteration loop.

4.2.3 Dependency of the sample size on the statistical measurements

Correctly used, the randomness in a stochastic analysis gives us a result that would demand enormous computing time with for example a factorial experiment design. Though, by choosing a finite number of samples is a stochastic analysis a rather rough approximation of the reality. In the case of large FE-models, as previously mentioned, the computer effort is a limiting factor and we can not run an arbitrary amount of simulations just to be on the safe side. This is why it is important to find out how many samples we actually need in order to achieve adequate results. How sensitive the result is to the sample size and what actually is meant with an adequate result differs depending the required result itself. The quantities that will be investigated here are the ones that in section 4.3.2 are recommended for evaluation of robustness for FE-models, i.e. *mean value* and *coefficient of variance* of the response function and *coefficients of correlation* and *rank correlation* of the input variables with respect to the response function.

In order to see how the different statistical quantities converge with increasing sample size a number of tests are performed.

The objective of the tests is to answer the following questions.

- Is the required sample size depending on which statistical quantity is chosen, i.e. if the mean value or the coefficient of correlation is the desired output?
- How does the required sample size differ with Monte Carlo Sampling and Latin Hypercube Sampling respectively?
- How the required sample size depend on the number of input variables?
- Is the demand regarding sample size different when the input variables have uniform distributions instead of normal distributions?
- How much does the character of the response function, e.g. linear or non-linear, monotonically or non-monotonically, affect the result?

The questions are investigated with an analytical function as well as with a simple non-linear FE-model with the objective to get a feeling of the needed sample size for a result with a certain safety.

Procedure of the tests

To answer the questions regarding statistical quantity, sampling method, number of variables and distribution of the input variables, initial tests were made with an *analytical function* as response function. By that we had full control over the behaviour of the response and it was easy to control the weighting of the input variables. The biggest advantage was the large amount of time saved in comparison to performing the tests with an actual FE-model. This way a very large number of test could be performed. The function of choice is the so called Rosenbrock-function

$$f(\bar{x}) = \sum_{i=1}^{n/2} 100 \cdot (x_{2i} - x_{2i-1}^2)^2 + (1 - x_{2i-1})^2 \quad (4.1)$$

also used as example in [ELDRED *et al.* 2001]. With very small alterations it can be used with an arbitrary number n of variables and it is easy to understand the relationships between the different variables.

The robustness of the function was evaluated with the use of Dakota, which performs the sample generation, and a Perl-script, that calculates the result, in this case the function value. The tests can be seen as an incomplete factorial experiment with three dimensions, *the sampling method*, *the number of variables* and *the distribution of the input variables*.

The following combinations were tested with the *analytical function*:

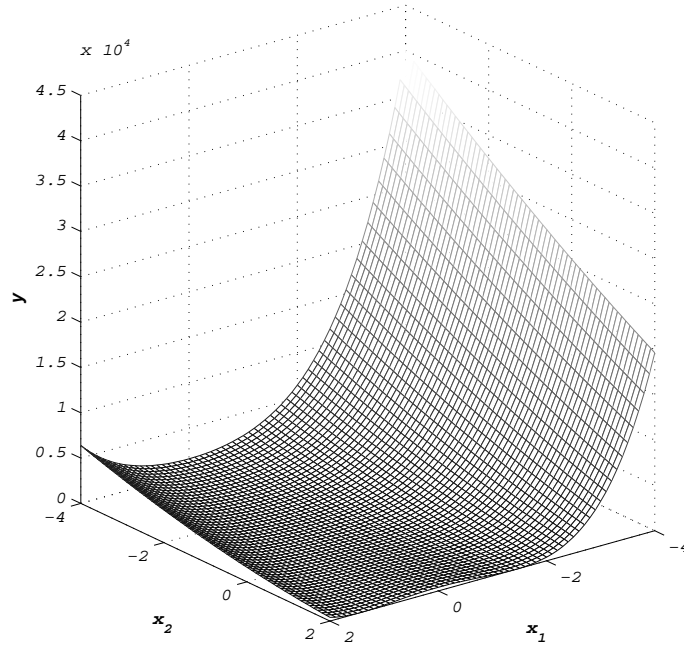


Figure 4.4: The Rosenbrock-function with 2 variables in the investigated interval

	Method	No. of vars	Distr.	Values
Test 1	PMC	2	Normal	$\bar{x}_i = -1, \sigma_x = 0.5$
Test 2	PMC	10	Normal	$\bar{x}_i = -1, \sigma_x = 0.5$
Test 3	PMC	100	Normal	$\bar{x}_1 = -2, \bar{x}_{i \neq 1} = -1, \sigma_x = 0.5$
Test 4	LHS	2	Normal	$\bar{x}_i = -1, \sigma_x = 0.5$
Test 5	LHS	10	Normal	$\bar{x}_i = -1, \sigma_x = 0.5$
Test 6	LHS	100	Normal	$\bar{x}_1 = -2, \bar{x}_{i \neq 1} = -1, \sigma_x = 0.5$
Test 7	PMC	10	Uniform	$x_i = [-2, 0]$
Test 8	LHS	10	Uniform	$x_i = [-2, 0]$
Test 9	LHS	100	Uniform	$x_1 = [-3, -1], x_{i \neq 1} = [-2, 0]$

The values are chosen so that the samples end up in an area where the function varies non-linear but smoothly and with a variance that is approximately equal to the variance that can occur in a robustness analysis with a real engineering problem.

The mean value of one of the input variables in test 3, 6 and 9 is different from the other. This is made to be able to compare the convergence rates for the tests with different number of variables. The insertion of additional variables in the Rosenbrock-function naturally changes the magnitude of the statistical quantities. The sum of the squared

correlation coefficients of all variables, with respect to one response, never exceeds 1 if the input variables are uncorrelated to each other. The Rosenbrock-function has the character that every second variable have equal influence on the result. In the tests with 100 variables, the 50 variables with equal influence will have a smaller correlation coefficient than in the tests with 10 variables. As mentioned in section 2.2.4, the variation of the coefficient of correlation depends strongly on the magnitude of the correlation coefficient. By doubling the mean value of x_1 and otherwise keeping the same distribution, a correlation coefficient of approximately the same magnitude as in the tests with 10 variables is achieved.

The deviation of the mean value and the coefficient of variance are not depending on their corresponding magnitudes. Neither the insertion of additional variables nor the small alteration of the mean of x_1 in the tests with 100 variables therefore affect the variance or the convergence rate of the mean value and the coefficient of variance.

To see if and how the variance of the statistical quantities decreases with an increases sample size, each test with the analytical function is performed with many different sample sizes. A robustness analysis is made with a sample size from 50 to 1000 with an increase of 50 samples for each step. In some cases the step in sample size is set to 100 to save time. This is only made after seeing that the variance converges very slowly and a step of 50 samples probably does not bring new information. For each sample size step, five different sets of samples was created to show the variance of the statistical measurements created by the randomness of the sample choice. The objectives of the tests with the analytical function are to see how the sample size affects the accuracy of the different statistical quantities and to see how the results vary because of the randomness.

How the convergence rate changes with another response function is investigated with a test with an actual *FE-model* denoted test 10. The test is performed with LHS and 10 uncertain variables with normal distribution. As a model a beam with unsymmetrical cross-section was chosen, one end is fully constrained and the other is loaded with a force F in negative z -direction. The parameters considered uncertain are the cross-section measurements $x_1, x_2, x_3, x_4, t_1, t_2, t_3$ and t_4 , the modulus of elasticity, E , and the load F , see figure 4.5. The distributions of the input variables are chosen so that they represent uncertainties that can occur in reality. The mean values and the standard deviations of the variables are presented in table 4.1. As response functions the maximum tensile stress in the cross section and the maximum displacement in y - and z -direction are evaluated. The material is considered linear elastic.

In this test the sample generation is made with Dakota and the calculations with the FE-solver ABAQUS [1]. The test was performed with sample sizes of 50 to 500 with an increase in sample size of 50 samples each time. At each sample size the test was repeated 5 times with newly generated samples.

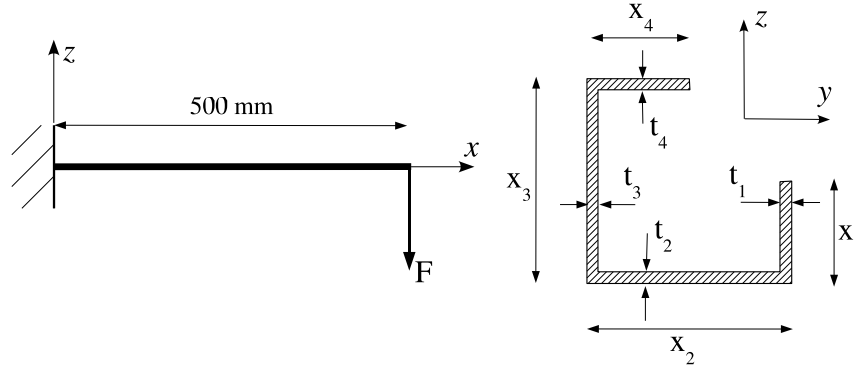


Figure 4.5: Loading and cross-section of the beam used as a simple FE-model.

	Method	No. of vars	Distr.	Values
Test 10	LHS	10	Normal	$\bar{x}_{1,4} = 15 \text{ mm}$, $\sigma_{x_{1,4}} = 1 \text{ mm}$ $\bar{x}_{2,3} = 30 \text{ mm}$, $\sigma_{x_{2,3}} = 2 \text{ mm}$ $\bar{t}_i = 1 \text{ mm}$, $\sigma_{t_i} = 0.1 \text{ mm}$ $\bar{E} = 210 \text{ GPa}$, $\sigma_E = 3 \text{ GPa}$ $\bar{F} = 500 \text{ N}$, $\sigma_F = 20 \text{ N}$

Table 4.1: The distributions of the input variables in Test 10

Results

Two things are evaluated in the tests:

1. Depending on the sample size, how well do the calculated statistical quantity represent the “real” value (with an infinite number of samples)?
2. How much can I rely on the calculated statistical quantities (how much do the quantities vary because of the randomness in the sample generation) with a certain sample size?

The “real” values of the quantities are approximated to be the mean value of the corresponding quantities of the 5 populations by a sample size of 1000. For the mean and the coefficient of variance, the largest deviation in each sample size from the “real” value divided with the “real” value is used as measurement of relative variation. When the variances of the correlation coefficients are to be compared, the difference of the maximum coefficient and the minimum coefficient in each sample size is used. Different combinations with different correlation coefficients r are evaluated in each test.

The results are presented in Tables 4.2, 4.3 and 4.4 and an example of the obtained diagrams can be seen in figure 4.6.

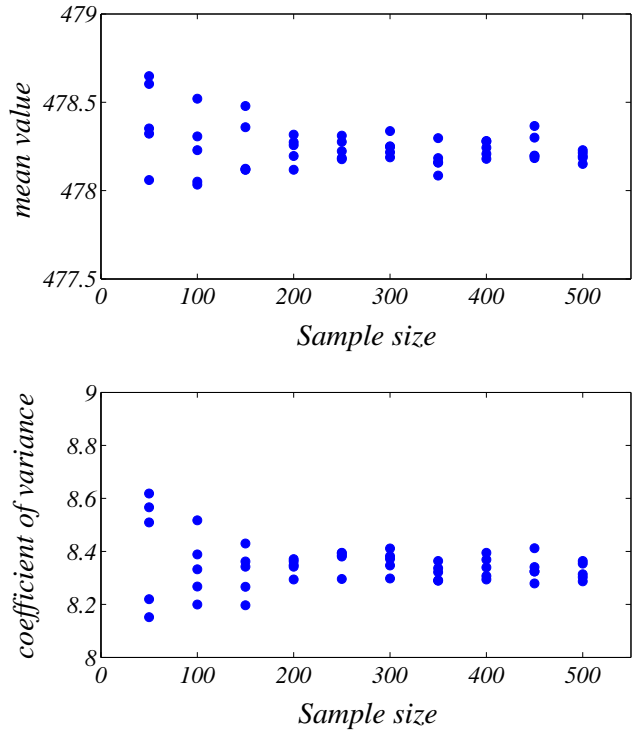


Figure 4.6: The different values of the mean value and coefficient of variance of the maximum tension stress obtained by a robustness analysis of a simple FE-model with different sample sizes.

	Sample size									
	50	100	150	200	250	300	350	400	450	500
Test 1	21.1	10.8	7.8	13.5	6.3	10.4	12.9	7.3	12.6	4.9
Test 2	8.2	7.0	5.3	5.6	5.0	6.1	2.6	2.6	1.9	2.6
Test 3	6.1	1.8	1.4	1.1	1.6	1.4	1.1	1.1	8.4	1.1
Test 4	7.6	1.2	2.7	1.1	6.1	0.5	0.5	1.5	2.6	1.0
Test 5	2.0	1.1	0.8	0.8	0.5	1.4	0.6	0.5	0.3	0.2
Test 6	0.3	0.4	0.1	0.1	0.3	0.1	0.1	0.1	0.1	0.1
Test 7	7.5	6.4	4.9	4.5	2.1	1.3	5.4	4.0	4.1	2.5
Test 8	0.8	0.3	0.5	0.1	0.4	0.1	0.1	0.3	0.2	0.2
Test 9	0.5	0.5	0.2	0.2	0.1	0.1	0.1	0.1	0.1	0.1
Test 10										
<i>max stress</i>	0.1	0.1	0.1	0	0	0	0	0	0	0
<i>displ. y</i>	0.3	0.3	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0
<i>displ. z</i>	0.1	0.1	0.1	0	0	0	0	0	0	0

Table 4.2: Deviation of the mean value of the response, in % of the “real” mean value.

	Sample size									
	50	100	150	200	250	300	350	400	450	500
Test 1	25.1	33.2	16.8	15.9	12.0	8.2	12.9	17.1	12.9	7.9
Test 2	13.4	17.0	16.1	5.2	7.8	7.3	8.6	14.2	8.8	6.5
Test 3	9.3	9.3	9.7	3.1	4.9	4.7	5.7	1.7	5.3	4.1
Test 4	21.8	5.3	16.3	7.3	46.4	3.9	4.2	8.5	19.5	4.3
Test 5	10.2	16.9	6.2	11.7	7.0	9.9	5.7	6.7	3.1	5.0
Test 6	14.5	8.4	6.6	4.4	3.2	3.1	4.6	3.4	3.6	3.0
Test 7	11.2	8.5	8.7	11.5	2.7	1.9	3.4	1.3	3.7	3.5
Test 8	8.7	4.3	4.3	4.2	5.8	3.0	6.1	2.9	2.8	3.9
Test 9	11.1	15.3	3.3	4.1	4.8	4.4	4.5	2.0	3.4	2.4
Test 10										
<i>max stress</i>	4.1	2.6	2.3	1.7	1.6	1.2	0.1	1.1	1.2	0.3
<i>displ. y</i>	5.6	4.6	2.1	2.9	3.2	3.0	1.6	1.2	3.2	1.9
<i>displ. z</i>	5.3	2.9	2.8	1.1	1.1	1.6	1.3	1.2	1.1	1.3

Table 4.3: Deviation of the coefficient of variance of the response, in % of the “real” coefficient of variance.

		Sample size									
		50	100	150	200	250	300	350	400	450	500
Test 1	r=-0.80	.08	.09	.04	.05	.07	.01	.04	.04	.05	.02
	r=-0.31	.24	.23	.02	.08	.08	.07	.09	.09	.03	.11
Test 2	r=-0.35	.45	.15	.29	.10	.03	.11	.10	.11	.06	.07
	r=-0.14	.12	.15	.07	.08	.12	.08	.07	.09	.06	.05
Test 3	r=-0.42	.40	.09	.11	.07	.08	.12	.12	.05	.05	.07
	r=-0.09	.10	.16	.05	.10	.05	.09	.01	.06	.08	.06
Test 4	r=-0.79	.06	.03	.04	.04	.10	.02	.02	.05	.06	.02
	r=-0.31	.15	.15	.09	.05	.19	.05	.06	.07	.10	.03
Test 5	r=-0.34	.09	.11	.05	.11	.05	.11	.10	.07	.07	.02
	r=-0.14	.14	.10	.08	.08	.05	.08	.08	.06	.05	.04
Test 6	r=-0.38	.08	.24	.06	.04	.05	.09	.10	.05	.04	.03
	r=-0.10	.11	.18	.19	.04	.16	.07	.07	.08	.02	.03
Test 7	r=-0.36	.13	.14	.06	.12	.08	.09	.03	.04	.03	.05
	r=-0.17	.26	.12	.14	.07	.10	.05	.05	.03	.07	.06
Test 8	r=-0.37	.14	.05	.05	.05	.06	.05	.06	.03	.06	.06
	r=-0.17	.13	.12	.04	.08	.09	.09	.05	.08	.06	.09
Test 9	r=-0.43	.38	.14	.06	.09	.05	.05	.06	.04	.02	.04
	r=-0.11	.31	.24	.03	.05	.03	.08	.03	.06	.05	.05
Test 10	r=-0.80	.08	.09	.04	.05	.07	.01	.04	.04	.05	.02
	r=-0.31	.24	.23	.02	.08	.08	.07	.09	.09	.03	.11
	r=-0.31	.24	.23	.02	.08	.08	.07	.09	.09	.03	.11

Table 4.4: The difference between the maximum and the minimum estimated coefficient of correlation of different variables regarding the responses, r is the mean value of the corresponding correlation coefficient.

Discussion

The sample size suggested in this chapter is only a recommendation. The required accuracy depends on the problem, so does also the computing cost of one simulation which result in a limitation in the sample size. Up to 10 000 samples are needed for statistical quantities that “exact” describe the reality. By accepting the uncertainty of the statistical values, much fewer samples can be used.

Required statistical quantity

The deviation of the coefficient of variance is in all cases larger than of the other quantities, compare tables 4.2, 4.3 and 4.4. Our definition of a robust design is based on its coefficient of variance (section 3.2) and it is important with a relatively high accuracy. By choosing a sample size that gives an acceptable reliability of the coefficient of variance, an acceptable accuracy of the other quantities is automatically achieved. The only quantity with higher variance is low correlation coefficients, they are however not of interest in a robustness analysis and a high variance is allowed. Furthermore is the variation of the correlation coefficients mostly depending on their magnitudes and not the samples size, see section 2.2.4.

Number of variables

When comparing tests 1-3 and 4-6 in Table 4.3 with each other, it can be noticed that the deviation of the mean values and the deviation of the coefficient of variance decreases with more variables.

Choice of sampling method

If tests 1-3 is compared with tests 4-6 it is easy to see that Latin Hypercube sampling has a large advantage to plain Monte Carlo. This difference is especially clear by the mean values. This originates from the fact that the means and variances of the input variables are much better represented with Latin Hypercube than with plain Monte Carlo, see figure 2.3. To get the same accuracy of the mean value with Monte Carlo as with Latin Hypercube after 200 samples not even 1000 samples are enough.

Choice of distribution

Comparing test 5 (normal distribution) with test 8 (uniform distribution) or test 2 with test 7, gives a clear result. The variance of the quantities are smaller with uniform distribution than with normal distribution. This result is probably not valid for all problems. As mentioned in the discussion of the influence of the number of variables, the Rosenbrock-function increases quadratic in the outer parts of the interval. The limits for the uniformly distributed variables are set rather tight, $\approx \bar{x} \pm 2 \cdot \sigma$. This reduces the response coefficient of variance and its variance in comparison to normally distributed variables that theoretically can assume values much further from the mean.

With other limits or for other responses, for example with a lot of irregularities, the result can be totally different.

In addition to the mean, the coefficient of variance and the correlation coefficient, the maximum and minimum response was investigated in the tests with uniformly distributed variables. To find these extreme response values it takes a lot of samples, even with 1000 samples the probability of coming close to the minimum and/or maximum is very small. It also highlights the uncertainty and thereby the cost of using a sampling method for a reliability analysis when the occasion to be analysed has a low probability of occurring.

The character of the response function

The three different response functions in the FE-model and the Rosenbrock-function with 10 variables will be compared, i.e. test 5 and test 10. To be able to completely evaluate the test with the FE-model, it is important to understand the response functions and their relationships with the uncertain variables. The maximum tension in the cross section is strongly dependent on the other two responses, the displacements in y- and z-direction (correlation coefficients of $r = -0.6$ and $r = -0.8$ respectively). The displacement in y-direction is linear correlated to x_1 by $r = 0.6$ and x_4 by $r = 0.6$ and the displacement in z-direction is linear correlated mainly to x_3 by $r = 0.7$ but also to F by $r = 0.5$. The other variables show much smaller correlations.

Looking in tables 4.2 and 4.3 one see that the deviations of the statistical quantities in test 10 are much smaller than in test 5. It is however difficult to draw any conclusions in this case. Both the variance of the mean value (the standard error of the mean) and the variance of the standard deviation are depending on the standard deviation itself. In this case is the standard deviation different in the different tests. The variation of the correlation coefficients still depends mostly on their magnitude.

Conclusions

It is difficult to draw conclusions from the tests since the mean value and the standard deviation of the different response functions are different. Clear is however the difference between using plain Monte Carlo and Latin hypercubes. A very big improvement of the variance of the statistical quantities is achieved with Latin Hypercube sampling, especially of the deviation of the mean values. The improvement of the deviation of the coefficient of variance is not quite that large but is still enough to choose LHS instead of PMC.

Which statistical quantities to be evaluated are also important for the choice of sample size. When it comes to determine if a model is robust or not, the coefficient of variance is the most important quantity. The coefficient of variance is also the quantity that varies the most due to the randomness in the sample generation. If we are satisfied with the accuracy of the coefficient of variance, then we are most certain satisfied with

the other quantities as well. Looking at tables 4.2, 4.3 and 4.4 it can be seen that the convergence rates for the statistical quantities are rather low. This means that we are going to accept a small uncertainty and let the limit of computing time control our choice of sample size.

The distribution of the variables do not seem to have direct influence on the required sample size. More influent is the character of the response function. This means that the required sample size is depending on the specific problem and it is difficult to recommend a general sample size for all models and objectives.

Another thing that came up during these test was the influence of outliers, both in the input variables and in the response. An extreme response value depends either on an extreme value of an input variable or on peaks in the response function. This phenomenon is of course interesting for the analysis but it is important that these outliers and their effect on the result are identified. It can be meaningful to remove them in the statistical calculations and only use them as a warning sign of what might happen in a production phase. Peaks are from highest interest but it is important to know how they affect the statistical values. Extreme input values are often of less importance since they normally do not exists in practise (due to for example quality controls in the production). In some cases it should even be considered to introduce bounds on the normal distribution in the sample generation. Bounds for normal distributions are possible in DAKOTA but it should be noticed that it changes the variance since the distribution then no longer is normal. With bounds on $2 \cdot \sigma$ the variance is reduced by 20%, with bounds on $3 \cdot \sigma$ the reduction is less than 1% and negligible.

4.2.4 Proposal for analysis method

The sampling methods, in particular Latin Hypercube sampling, are well suited for robustness analyses of FE-models. Very few demands are set on the model and the art of simulation. The only disadvantage is that they often require that the simulation is repeated several times which can lead to long computing times. The computing time can be reduced through the possibility of parallelisation. A robustness analysis with LHS can be made on any problem as long as the available computer power or computing time is sufficient. A sampling method with Latin hypercubes is the recommended analysis method in this report.

If the computation time allows it, a sample size of 200 is recommended. With an additional search for outliers and evaluation of how these outliers affect the statistical quantities, then the accuracy in the statistical quantities should be enough for most problems. A sample size of 200 also gives a little room for later analyses where samples are modified and removed. If it, for time reasons, is not possible to run 200 simulations then it is possible to reduce the sample size. This puts higher demands on the following evaluation and a reduction must be done with care. The statistical quantities are then

more uncertain and must be interpreted with more scepticism. Moreover, it is more risky to remove samples in the evaluation and therefore harder to see all relationships and to find suitable actions for improvement.

4.3 Recommended method for data evaluation

The data from a stochastic robustness analysis can give answers to many questions if evaluated correctly. Different methods and statistical quantities which can help the engineer in the evaluation exist. The evaluation of mean value and variance is simple. Straight-forward mathematical functions calculates the desired statistical quantities. The evaluation of relationships between variables is more complex. In section 2.2, two methods for this kind of data evaluation are described, *regression analysis* and *analysis of variance, ANOVA*.

4.3.1 Suitability of the available methods

Analysis of variance, ANOVA

The disadvantage with *ANOVA* is that it demands samples in a certain pattern and is by that badly suited for stochastic analyses. Since the choice of analysis method for the robustness analysis is a stochastic sampling method using Latin hypercubes, ANOVA is a bad choice as evaluation method in a robustness analysis.

Regression analysis

The *regression analysis*, on the other hand, suits well together with stochastic sampling methods. No specific demands are set on how the samples are chosen and the calculations are relatively easy. The disadvantage of the method is the uncertainty created by the randomness. Two sample populations with same sample size can give different statistical quantities. If the samples are generated following suitable rules, as in for example LHS, and containing a suitable sample size, this uncertainty is acceptable.

The big disadvantage with a regression analysis is the dependency of the predefined regression model (linear, quadratic etc.). With the wrong model is the result after the analysis useless. The suitability of the defined regression model must be verified.

A regression analysis should be followed by a correlation analysis. The correlation analysis after a robustness analysis is executed to find input variables with high influence on the response. The regression analysis approximates a curve that represent a possible relationship. It does however not show how well the samples fits to this curve. The correlation coefficients show how far away from the curve the samples are located. Samples widely spread around the curve indicates that other variables are influent on the response.

Scatter plot analysis

The most easy way to see if a relationship exists or not is to look at the samples in for example a scatter plot. It is time consuming to look at every plot when many combinations of variables are to be evaluated but it is definitely the most secure way. Even for someone totally without statistical knowledge it is easy to see every kind of relationship between two variables in a scatter plot. Possible extreme values, owing to very rare combinations or failed simulations, are easy to detect in a plot and can be removed if necessary. Clusters can also be detected. If it is made possible to easily go through a large number of plots in a short period of time, scatter plots are much more informative than statistical quantities in the case of a robustness analysis. Optimal is of course if the scatter plots and the statistical quantities are used together.

Different kinds of plots are also very useful when it comes to taking the analysis a step further and answer the question 'what must we do to improve the robustness of the model?'

4.3.2 Recommended statistical quantities

The *mean value* is of course an important quantity in robustness analyses. It is essential that the mean value coincides with the most desired value.

For the quantification of variation, the *coefficient of variance* is recommended. It presents the standard deviation in percent of the mean value and is more informative than the standard deviation itself. A standard deviation of 20 is small when the mean value is 10 000 but not acceptable if the mean value is 30. The same values for the coefficient of variance are 0.2% and 67% respectively. Important to notice is that the coefficient of variance gets infinitely large when the mean value is zero. This can be very misleading, depending on how the parametrisation of the model is made. A correction of the mean in the evaluation phase can be necessary for a useful coefficient of variance.

Since the recommended choice of evaluation method is regression analysis and a following correlation analysis, the *coefficient of correlation* is the recommended quantity of relationship between variables.

It is important to be aware of the relatively large confidence intervals of the correlation coefficients. The confidence interval of a correlation coefficient of a totally random variable (samples chosen with PMC) can be calculated with the statistics of Fisher's z -transformation, see section 2.2.4. The confidence intervals for correlation coefficients with samples generated with LHS can be derived experimentally. For an estimated correlation coefficient of 0.3 and a sample size of 100 created with plain Latin Hypercube, the 95% confidence interval is 0.1. This means that the actual correlation coefficient is, with 95% probability, a value between 0,25 and 0.35. With plain Monte Carlo the convergence intervals are a lot larger. For an estimated correlation coefficient of 0.3

and a sample size of 100, the 95% confidence interval is 0.37. For an estimated coefficient of correlation of 0.7 the confidence intervals for LHS and PMC are 0.04 and 0.2 respectively [BUCHER 2005A]. In the previous chapter, Latin Hypercube sampling was recommended as analysis method. The confidence intervals for a correlation coefficient above 0.5 with LHS is smaller than 0.07 as long as the sample size is larger than 100 and it is not necessary to calculate the confidence interval as long as the coefficients are used more as indicators and not exact measurements.

The coefficients of correlation is only a fair measurement as long as the approximated curve really fits to the real dependency. A linear correlation coefficients gives a poor direction of the dependency when the relationship in reality is quadratic. This means that it can be very dangerous to rely on a low correlation coefficient. To guarantee that all relationships are well represented in correlation coefficients a very large number of regression analyses is necessary. It is informative to look at the coefficient of determinations (section 2.2.4) after a correlation analysis, for example a linear. If the sum of the coefficients of determination is close to 100%, then almost all response variation is explained by the correlation coefficients obtained in the correlation analysis. If not, another correlation analysis, for example a quadratic, should be tested.

To cover possible additional relationships it is recommended to extend the correlation analysis with the *rankcorrelation coefficient*. It is easy to calculate and may find relationships, the correlation analysis have failed to find.

An alternative to the correlation analysis of higher degree is to divide the variable interval in smaller parts and calculate the linear correlation in each of these intervals separately. It is not unusual in physical models that a variable has practical no influence up to a certain limit and then is the most influent. This kind of relationship will probably be well represented with a exponential regression analysis but would also be noticed by a linear regression analysis in smaller intervals. The same holds for a perfect cubic curve, if the interval is divided in three parts, the relationship would be well represented by a linear regression analysis. Adding the feature of dividing the interval into smaller parts, generalises the linear regression analysis. A warning must be issued for using this correlation coefficients together with normal distributed variables. The confidence interval of the correlation coefficients are depending on the samples size, see section 2.2.4. With a normal distribution, the outer parts of the interval might contain to few samples for a reliable correlation analysis. For the same reason, it is dangerous to divide the interval in too many parts, two or three parts are generally enough.

4.3.3 Proposed evaluation procedure

The proposed procedure for the evaluation of data from a robustness analysis with a sampling method can be summarised as:

1. Evaluate the mean and the variance of the response functions.
2. Perform correlation analyses to find combinations of variables which are related.
3. Investigate these combinations more thoroughly in scatter plots to see how the relationship actually looks like.
4. Ideally, go through the other combinations quickly to find possible relationships that were not discovered by the correlation analyses.
5. By modifying the samples, find a suitable way to improve the robustness of the model.

The first and most important thing is to establish if the design is robust or not. This is made by analysing the histograms, means and coefficients of variance of the output. It is necessary to study a histogram or something equivalent since the variance has different meanings for different distributions. If the input variables are chosen correctly and have been assigned the right distribution, and the response does not vary more than allowed, then the model is robust. This presumes, of course, that the underlying deterministic simulation model is accurate for ideal conditions.

If the variance is too large or the mean value is badly located then the reason must be found. If the number of variables is small, it is recommended that a linear and a quadratic correlation analysis as well as a rank correlation analysis is made. These three analyses are quickly executed, and are usually able to find all important relationships. Their exact characteristics can be further investigated in a scatter plot. Ideally, all plots should be closely investigated but it is in most cases enough to investigate the combinations with a correlation coefficient above 0.5. The sum of the coefficients of determination for each response should be close to 100%. If not, some part of the response's variance is caused by something else and it is recommended that a cubic correlation analysis or a correlation analysis with the data divided in smaller parts is performed.

If the number of variables is large, then the evaluation can be more complex. The first aspect is that the calculation of the statistical quantities needs more time. The second, and main, aspect is that it is difficult to identify correlations when many variables have equally large influence on the response. In the case of a large number of variables it might be necessary to lower the definition on an important relationship to for example 0.4. Remember that the confidence intervals i.e. the uncertainty of low correlation coefficients are larger than of high correlation coefficients. The important parameter combinations must be very closely analysed to find out what can be done to improve the robustness of the model. This is preferably made with different plots. A good way is to identify samples with especially high or low response function and then compare the input values of these samples to find possible patterns. Another efficient method is to reduce the variance of one variable (remove the samples in the outer parts of the interval) and see how that affects the correlation coefficients of the other variables. The latter method demands a rather large number of samples since the confidence interval

for the correlation coefficients increases with a smaller number of samples.

All of the above explained evaluations steps are supported by the special tool developed within this thesis and described in section 5.1.

Chapter 5

KNUT - A Matlab-based tool for statistical evaluation

5.1 Tool for an efficient statistical evaluation

A necessity for an efficient evaluation is a quick way to produce plots and calculate the statistical quantities. It is important to quickly be able to identify single or groups of samples for a closer investigation. In the tests in section 4.2.3 it was for example noticed that single, very extreme values can have unwanted, strong effects on the statistical quantities.

It was noticed during the work that a tool that fulfils our demands on the evaluation procedure is hard to find. It was instead decided that a tool should be developed, that have the required qualities and works well together with the softwares we use in this kinds of analyses, for example DAKOTA.

5.1.1 Demands

The demands set on the tool are

- a simple interface that helps the user through the evaluation.
- ability to use different kinds of input files, i.e. the output ASCII-file from DAKOTA or an Excel-file, without extra interfaces.
- ability to calculate all the statistical quantities recommended for the evaluation in a robustness analysis.
- ability to produce different plots for further evaluation of the data.
- ability to produce plots with appearance suitable for usage in presentations.
- ability to quickly identify one or more samples.
- ability to mark and remove samples in the plots in an easy and quick way to simplify the evaluation.
- possibility to link different plots. Changes in one plot shall be seen in other plots as well to be able to find indirect relationships and to determine consequences of a modification.

The main focus is the comprehensibility and rapidity. It is a wish that the tool has a graphical interface and that it easily can be extended for further analysis, e.g. the data analysis after an optimisation.

5.1.2 The structure of the tool

It was decided that the tool was to be made in MATLAB [4] for the following reasons.

- Functions for the calculation of some of the statistical quantities are available in MATLAB and with small programming costs the regression analysis with the following correlation analysis can be made.
- The codes for most of the plots we need already exist and contain helpful features as zoom and rotate. The appearance of the plots are easily modified to fit the users specific wishes. This means that most of the work could be concentrated on customising the functions to fit our needs exactly.
- MATLAB contains any easy way to create GUIs (graphical user interfaces). A GUI is a good way to increase the comprehensibleness for user without MATLAB experience. In the GUI all the choices can be made with button and mouse clicks.

The use of MATLAB meant that most of the work could be concentrated on customising the functions to fit our needs exactly. As a possible further step in the development of the tool, the tool should be able to run without MATLAB.

The tool is divided in two parts that can be used either separately or together. The first part is the statistical part, that is designed for robustness analyses and parameter studies. Here a regression analysis is made and the recommended statistical quantities, mentioned earlier, are calculated. This part also includes an extensive plot analysis where it is possible to modify the data and immediately see how the statistical quantities are affected.

The second part creates plots e.g. interpolated response surfaces, suitable for different graphical analyses and for presentations, in a simple and user-friendly way. This part is more general and can be used for other analyses than robustness evaluations for example parameter studies and evaluation of an optimisation.

The samples are divided in five groups: active, marked, removed, marked by constraints and removed by constraints. These are kept in the entire evaluation. If one sample is marked in one plot, then this sample will stay marked in all other following plots until its state is manually changed.

5.1.3 Possibilities and recommended areas of use

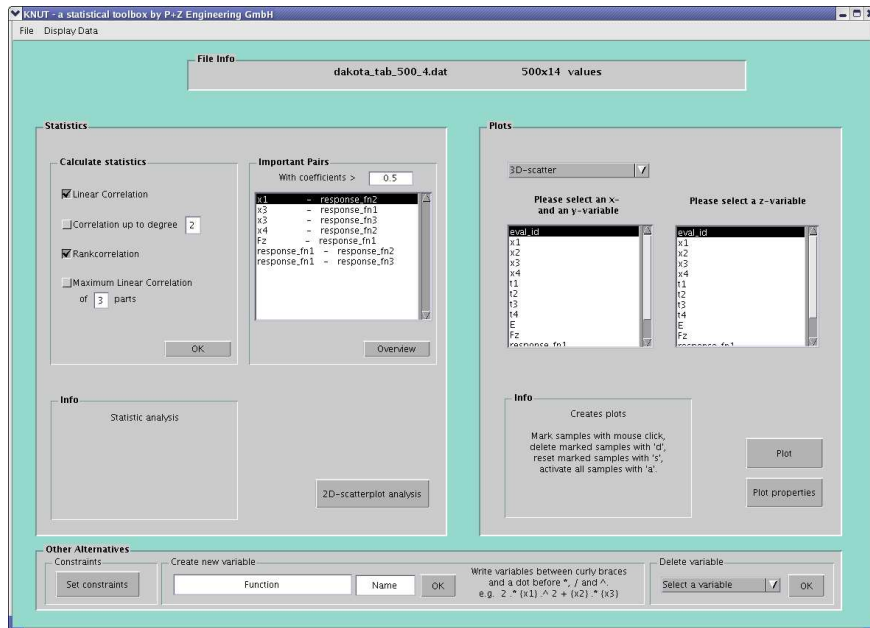


Figure 5.1: The main page in the statistical toolbox KNUT

1. File menu

- Reads files in .dat-, .csv- and .mat-format.
- Saves the current session in the MATLAB file format .mat.
- Saves the calculated statistical data in a text file.

2. Display Data menu

Lists the sample values separated into

- Active Data
- Marked data
- Removed Data

In the lists it is possible to activate, mark or remove individual samples.

3. File info

In the file info box, two things are displayed.

- The file name.
- The number of values in the file.

4. Statistics

Choose the statistical data to be calculated.

- Linear correlation
- Correlation after a regression analysis with polynomials up to the defined degree.
- Rank correlation
- The variable interval is divided into a defined number of parts. In each part, the linear correlation is calculated and the maximum correlation coefficient of the parts are returned.

The calculated statistics are based on all samples. To calculate statistics for a part of the samples, use the 2D scatter plot analysis.

5. Important pairs

- Important pairs of parameters, with a coefficient of one or more of the correlations that exceed the defined limit, are displayed in the list.
- For a quick overview mark the pairs of interest and press 'overview' to plot the chosen important combinations.

6. 2D scatter plot analysis

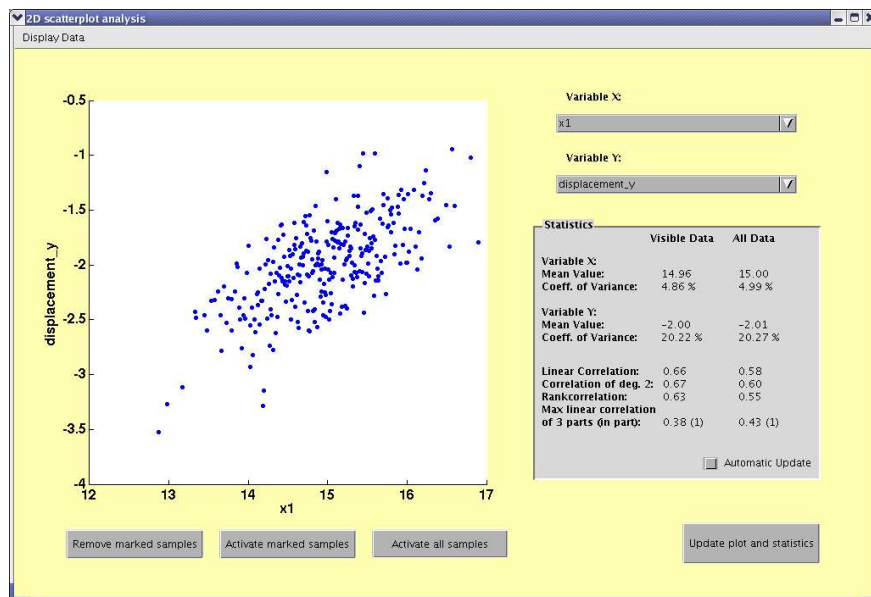


Figure 5.2: The interface for a thorough statistical evaluation of data in KNUT

- All possible combinations of variables are available for a closer analysis.

- All calculated statistical quantities are displayed for the actual combination. One column contains the statistics for all originally loaded samples and one column contains the statistics for the samples visible in the plot.
- In the plot it is possible to mark samples with the mouse. The marked samples can be removed with 'd' or by pressing 'Removed marked samples' and again displayed with 'a' or 'Activate all samples'. To activate only the marked samples and not the removed, press 's' or 'Activate marked samples'.
- If samples were marked or removed in another plot, a mouse click in this window updates the plot.
- To calculate the statistical after a modification, press 'Update plot and statistics'. Alternatively, check 'Automatic Update' and the calculations are made automatically after a modification in the plot.

7. Plots

- Histogram
- Plotmatrix - In the plotmatrix it is possible to mark samples with the mouse. The marked samples can be removed with 'd', and again activated with 'a'. To activate the marked samples only, press 's'. The other plots in the plotmatrix are automatically updated.
- 2D scatter plot - Samples can be marked in the same way as in the plotmatrix.

By marking several variables as x-variables, multiple plots are displayed in the same window. By selecting 'Add in previous plot', it is possible to get plots with different y-variables in the same window.

- 3D scatter plot
- Surface plot with interpolated values - From the samples a response surface function is approximated with cubic interpolation.

With 'Add in previous plot' the new surface is added in the same axes. This can be useful in a "before and after"-comparison.

- Surface plot with exact values - This plot demands that the samples are evenly distributed, for example after a full-factorial parameter study.
- Contour plot - A three dimensional plot with the z-axis directed out from the plot and the values are represented by contour lines.
- Barplot - The magnitude of the correlation coefficients of all or the selected variables with respect to a selected response are plotted for a clear overview.

In all the plots it is possible to change colours, fonts, ticks etc. as in a normal figure in MATLAB. Other functions as zoom and rotate are also still available. The plots can be saved in a number of different formats for use in reports or presentations.

If samples are marked or deleted in one plot, the other plots are updated with a mouse click in the corresponding window.

8. **Plot properties**

The most common properties of the plot, for example font size and marker colour can be modified in this special GUI. The individual settings can be saved and reloaded in another session to quickly obtain the desired property settings.

9. **Constraints**

Under constraints it is possible to mark or delete samples that exceed defined limits in one or more variables or responses. The samples marked in constraints are still considered active.

10. **Create new variable**

It is also possible to create a new variable as a function of the existing variables. Creating a variable as a function of the old, e.g. $x_3 = x_1 \cdot x_2$ can bring new knowledges about relationships.

11. **Delete a variable**

By choosing from a list with the existing variables, a variable can be deleted. Removing a variable where it is already established that no influence exists, can simplify the rest of the evaluation.

Chapter 6

Examples

6.1 Robustness analysis of a simple analytical function

This example is made to highlight the consequences of the choices made during a robustness analysis. A simple analytical function $y = x_1^2 + x_2$ is evaluated. Only two variables are present and it is easy to see their influence on the function value y . The mean value of the two input variables are zero, but they are uncertain and can assume other values. The objectives of the robustness analysis are:

- How does the function value vary because of the uncertainty of the input variables?
- Which of the two input variables is most responsible for the variation of the function value?
- What is the best way of improving the variables so that the function value ends up in the interval $[-1.5, 1.5]$ with 99.7%-probability.

6.1.1 Choice of variables

The first choices to be made in a robustness analysis are which input variables to be included in the analysis and which distributions best represent their uncertainties. To see the consequences of the choices, different combinations of distributions will be analysed.

	x_1		x_2	
1.	normal	$\sigma = 0.5$	constant	$x_2 = 0$
2.	constant	$x_1 = 0$	normal	$\sigma = 0.5$
3.	normal	$\sigma = 0.5$	normal	$\sigma = 0.5$
4.	normal	$\sigma = 0.7$	normal	$\sigma = 0.7$
5.	uniform	$[-1.5, 1.5]$	uniform	$[-1.5, 1.5]$

Table 6.1: 5 different combinations of input variable distributions

6.1.2 Sample generation

A robustness analysis of y is executed with 200 samples generated with LHS. The sample generation is made with DAKOTA and the calculation of the function value is made with a Perl-script. DAKOTA produces a text file with the different combinations of input values and their corresponding function values.

6.1.3 Data evaluation

The evaluation begins with an analyse of the mean and variance of the function value. The mean values, standard deviations and coefficients of variance of y for the different combinations of distributions are displayed in table 6.2. An increase of the mean leads to a decrease of the *coefficient of variance*. To be able to keep the samples within the desired limit is a small mean value preferable. The demands on the standard deviation are higher with a high mean value. The coefficients of variance is therefore a bad measurement for the variation of y in this example. Variable combination 5 has a significant larger standard deviation and a larger mean value of y than combination 3 and is by that less robust. The coefficient of variance of y with variable combination 5 is smaller than with variable combination 3.

For a complete understanding of how the function values vary, a histogram is a good complement to the statistical quantities.

	mean value	standard deviation	coefficient of variance
1.	0.25	0.34	137%
2.	0.0	0.15	- ¹
3.	0.26	0.63	246%
4.	0.49	1.02	208%
5.	0.75	1.07	143%

Table 6.2: The mean value, the standard deviation and coefficient of variance of y for the 5 different combinations of input variable distributions.

The function value is zero when the input variables are in their mean values respectively. Normally, the model is so defined that the mean values of the input variables lead to the most desired system response. The mean function value is because of the uncertainties in the input variables only zero in the case where x_1 is constant. This indicates that x_1 is the variable responsible for the displacement of the function mean.

The variance of the function value is too large to fulfil the goal of 99.7% of the samples within $[-1.5, 1.5]$ in all the cases where both variables are considered uncertain, see the histograms in figures 6.1, 6.2, 6.3, 6.4 and 6.5. In each of those cases, several samples have function values above 1.5. It is desired to find the input variable/variables responsible for the variation to find an improvement strategy. The decision in section 2.2 is to use a regression analysis with an adjacent correlation analysis.

¹the coefficient of variance is useless when the mean value is zero, see section 2.2.4, the standard deviation is a better measurement of variation in these cases

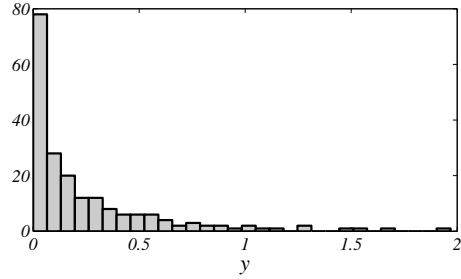


Figure 6.1: Histogram of y with input combination 1, x_1 normally distributed, $\sigma = 0.5$, and x_2 constant.

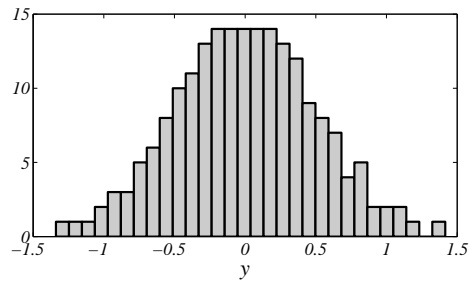


Figure 6.2: Histogram of y with input combination 2, x_1 constant and x_2 normally distributed, $\sigma = 0.5$.

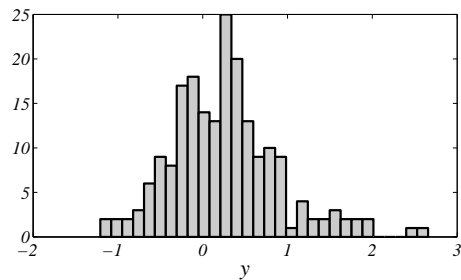


Figure 6.3: Histogram of y with input combination 3, x_1 and x_2 normally distributed, $\sigma = 0.5$.

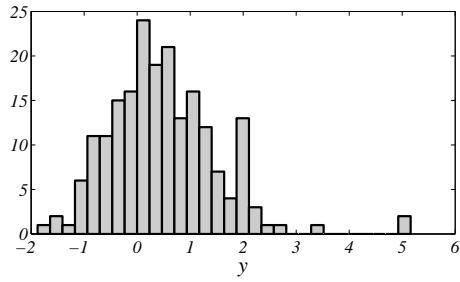


Figure 6.4: Histogram of y with input combination 4, x_1 and x_2 normally distributed, $\sigma = 0.7$.

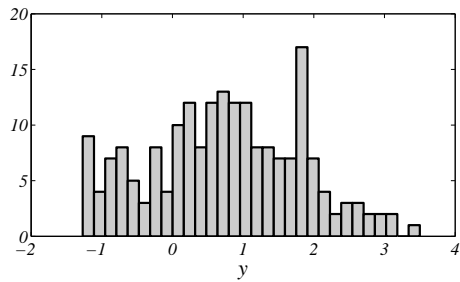


Figure 6.5: Histogram of y with input combination 5, x_1 and x_2 uniformly distributed.

The regression analysis fits a mathematical function with predefined character, for example linear, to the samples. This can be useful for the understanding of the character of the relationship and can be helpful when deriving measures of improvement for the robustness of the design. It is also possible to derive which variable causes the displacement of the mean with help of the regression coefficients. But it does not tell how well the samples fit to this mathematical function. By that it is difficult to say which of the two variables is the most influent on the variation of y with a regression analysis alone. It is also difficult if the defined function characteristic is a good representation of the actual relationship. A linear regression model represents a quadratic relationship poorly. Table 6.3 shows the approximated regression curves \hat{y} after a linear and a quadratic regression analysis. Both the linear and the quadratic regression analysis are made with each of the input variables separately. The models used are

$$\hat{y} = \beta_1 x_i + \beta_0 \tag{6.1}$$

and

$$\hat{y} = \beta_2 x_i^2 + \beta_1 x_i + \beta_0 \tag{6.2}$$

where x_i is one of the variables and β_j is a regression coefficient.

	linear regression	quadratic regression
1.	$\hat{y} = -0.0137x_1 + 0.2476$	$\hat{y} = x_1^2$
2.	$\hat{y} = x_2$	$\hat{y} = x_2$
3.	$\hat{y} = -0.0282x_1 + 0.2593$ $\hat{y} = 0.9832x_2 + 0.2596$	$\hat{y} = 0.9757x_1^2 - 0.0107x_1 + 0.0061$ $\hat{y} = 0.0281x_2^2 + 0.9834x_2 + 0.2525$
4.	$\hat{y} = -0.0564x_1 + 0.4932$ $\hat{y} = 1.0646x_2 + 0.4936$	$\hat{y} = 1.0643x_1^2 + 0.0116x_1 - 0.0319$ $\hat{y} = -0.0493x_2^2 + 1.0642x_2 + 0.5178$
5.	$\hat{y} = 0.0124x_1 + 0.7495$ $\hat{y} = 0.9616x_2 + 0.7497$	$\hat{y} = 0.9360x_1^2 + 0.0122x_1 + 0.0478$ $\hat{y} = 0.0443x_2^2 + 0.9616x_2 + 0.7165$

Table 6.3: The approximated regression curves for the different input variable combinations.

The coefficients of correlation of x_1 in table 6.4 show how misleading a linear regression analysis can be. This is detected by the sum of coefficient of determination R^2 . After the linear regression analysis only 4% of the variation of y in case 1 is explained. Both variables are uncorrelated to their squares since the mean values are zero. It is therefore possible to calculate R^2 as the sum of the squared correlation coefficients with the quadratic regression model as well, see the explanation on R^2 on section 2.2.4. Around 100% of the variation of y in all five cases is explained after the quadratic analysis. In case 1 and 2, where the variables act alone, are the correlation coefficients 1.0

with the correct regression analysis. When both variables act together, the correlation coefficients decrease. If additional variables with equal influence are added to the function, then the correlation of each variable decreases even more.

With distribution combination 3, x_2 is slightly more influent than x_1 . In case 4, where the variables have larger standard deviation, the difference between the influences of the two variables seems to decrease. This is intuitive correct when thinking of the characteristic of the function. Notice how when one coefficient increases the other must decrease in order to keep the sum of coefficients of determination below 100%.

The rankcorrelation does not bring extra information in this example, see table 6.5. Neither does the partial correlation coefficient. The maximum correlation coefficient of three parts is more useful. It can, instead of the quadratic correlation analysis, detect other relationships than linear. It has in this case found a nearly linear relationship with y in the first and last part of x_1 , see figure 6.6. Calculating the correlation with the data divided in multiple parts with normally distributed data must be made with care, see section 4.3.2. With distribution combination 3, 35 samples are located in the first part and only 16 samples in the third part. These sample sizes give for a correlation coefficient of 0.7 confidence intervals of approximately 0.07 and 0.15 respectively. For an estimated correlation coefficient of 0.5 the confidence intervals are approximately 0.13 and 0.25 respectively [BUCHER 2005A]. For an explanation of confidence intervals, see section 4.3.2. The problem of large confidence intervals is seen by comparing case 3 with normally distributed variables with case 5 with uniformly distributed variables. In case 5, a third of the samples are located in the outer parts and the correlation coefficients are distinctly smaller.

For a better understanding of the coefficients is it helpful to compare them with their corresponding scatter plot. The function is displayed in figure 6.7 and the scatter plots from the different input variable combinations are displayed in figures 6.8, 6.9, 6.10, 6.11 and 6.12.

	$\mathbf{r}_{x_i y_{lin}}$		$\mathbf{R}_{x_i y_{lin}}^2$	$\mathbf{r}_{x_i y_{quad}}$		$\mathbf{R}_{x_i y_{quad}}^2$
	x_1	x_2		x_1	x_2	
1.	-0.02	-	4%	1.00	-	100%
2.	-	1.00	100%	-	1.00	100%
3.	0.02	0.77	59%	0.63	0.77	99%
4.	-0.04	0.73	53%	0.73	0.73	107% ²
5.	0.01	0.78	61%	0.59	0.78	96%

Table 6.4: Correlation coefficients of the input variables with respect to y after linear $r_{x_i y_{lin}}$ and quadratic regression analysis $r_{x_i y_{quad}}$ with their corresponding sums of coefficients of determination $R_{x_i y_{lin}}^2$ and $R_{x_i y_{quad}}^2$.

	$\mathbf{r}'_{x_i y}$		$\mathbf{r}_{x_i y}^{\max}$		$\mathbf{r}_{x_i y x_{j_{lin}}}$	
	x_1	x_2	x_1	x_2	$x_1 y x_2$	$x_2 y x_1$
1.	0.00	-	-0.99(1)	-	-0.02	-
2.	-	1.00	-	1.00(1)	-	1.00
3.	0.00	0.85	-0.77(1)	0.59(2)	0.04	0.78
4.	0.01	0.78	0.81(3)	0.60(1)	0.08	0.72
5.	0.01	0.78	0.57(3)	0.32(2)	0.02	0.78

Table 6.5: Rankcorrelation coefficients $r'_{x_i y}$, maximum correlation of three parts $r_{x_i y}^{\max}$ and partial correlation coefficient $r_{x_i y | x_{j_{lin}}}$.

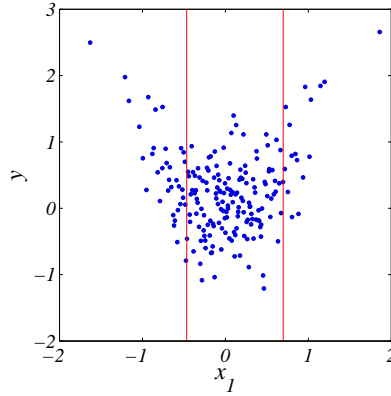


Figure 6.6: Scatter plot of y and x_1 in combination 3 with the limits for the maximum correlation in three parts displayed.

²the sum of coefficients of determination R^2 sometimes exceeds 100%, this dues to small correlations between input variables and to the confidence intervals of the correlation coefficients

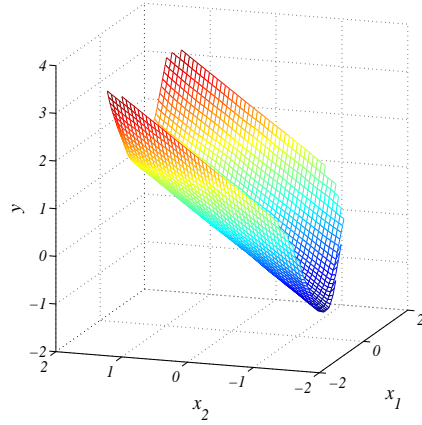


Figure 6.7: A response surface of the function $y = x_1^2 + x_2$ created by cubic interpolation between the samples.

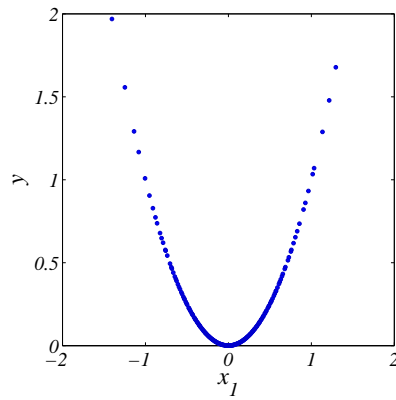


Figure 6.8: Scatter plot of y and the input variables in combination 1.

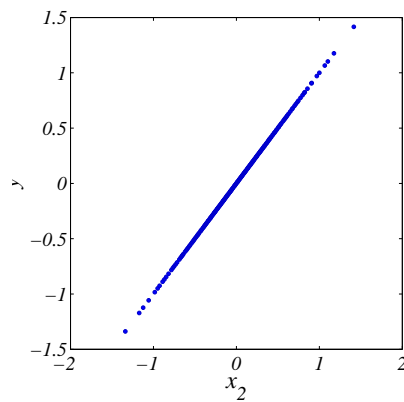


Figure 6.9: Scatter plot of y and the input variables in combination 2.

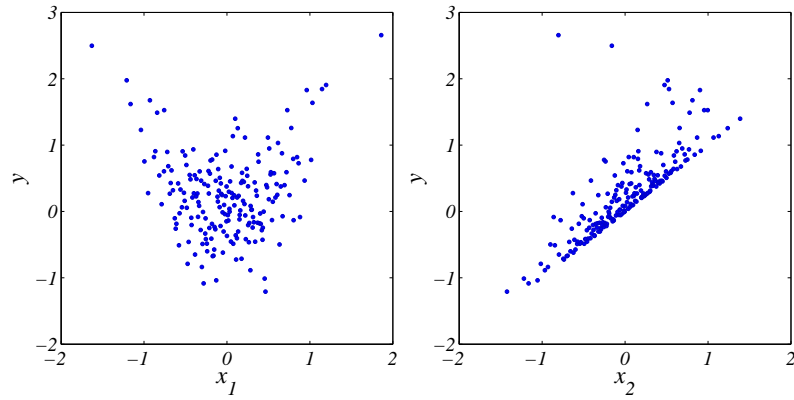


Figure 6.10: Scatter plot of y and the input variables in combination 3.

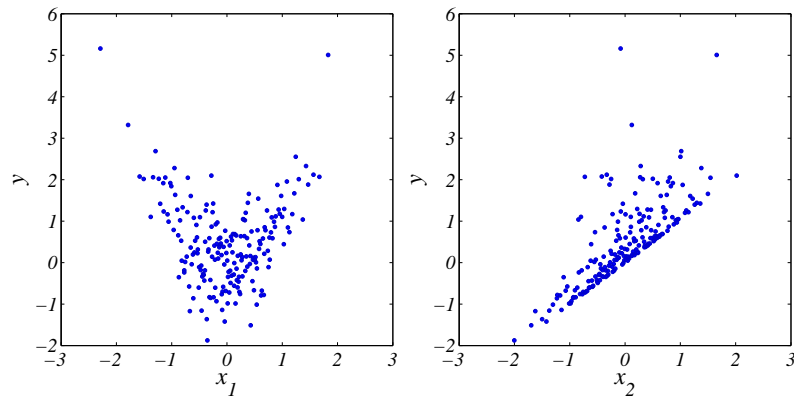


Figure 6.11: Scatter plot of y and the input variables in combination 4.

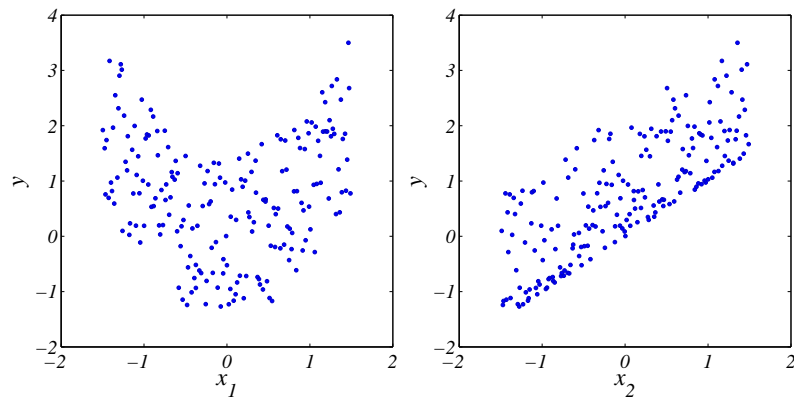


Figure 6.12: Scatter plot of y and the input variables in combination 5.

6.1.4 Improvement proposal

When looking for an improvement strategy, studying the different plots is the quickest and most informative way. Here a strategy for a decrease of the mean value and the variance of y with input combination 3 (normally distributed variables with $\sigma = 0.5$) will be proposed. The goal is to come below 1.5 with 99.7% of the samples. The function value with input variable combination 3 is approximated to be normally distributed and we can approximate the goal as

$$\bar{y} \pm 3 \cdot \sigma_y \leq 1.5 \quad (6.3)$$

The samples marked with stars in figure 6.13 are causing the increase of the mean value. The plots shows that the best way to avoid these high function values is to decrease the variance of x_1 . Note that the correlation coefficients indicate that x_2 is the variable with the strongest influence. We have two alternatives in decreasing the variance of x_1 . Either limits are set at -1 and 1 respectively (in reality with for example a quality control) or the variance is reduced to approximately 0.33 (realisable with for example an improvement of the production process). By simply removing the samples with x_1 smaller than -1 or larger than 1 in the original sample population the statistics are changed. With the old values in brackets, the statistical quantities are now:

mean value \bar{y}	0.19	(0.26)
standard deviation σ_y	0.56	(0.63)
correlation $r_{x_1 y_{quad}}$	0.40	(0.63)
correlation $r_{x_2 y_{lin}}$	0.91	(0.77)

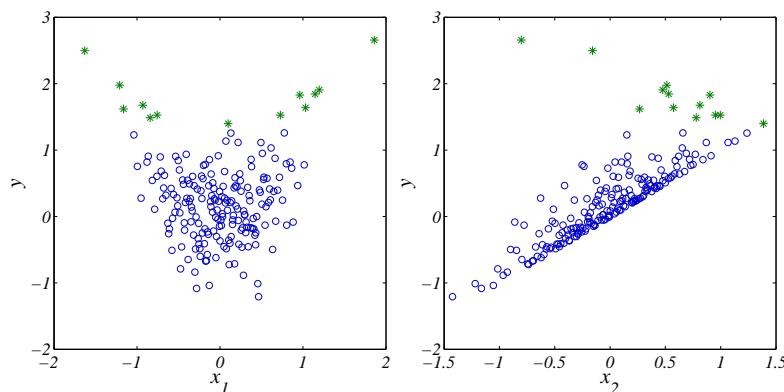


Figure 6.13: Scatter plot of y and the input variables for input variable combination 3. The samples with function value above 1.3 are marked with stars. Removing them gives a mean value of y of approximately 0.

We have achieved an improvement i.e. a decrease of both the mean value and the variance. Note that this improvement of the mean value is much more difficult to perform by a reduction of the variance of x_2 even though its coefficient of correlation is higher. The correlation coefficients have changed as well after the reduction of the variance of x_1 . x_2 is now the variable with strongest influence.

To reach the desired goal, the standard deviation must be reduced to at least $0.43 \cdot ((1.5 - \bar{y})/)$. It is not enough to reduce the variance of x_2 with the same magnitude as the reduction of variance of x_1 . Removing all samples with x_2 smaller than -0.75 or larger than 0.75 , marked with triangles in figure 6.14, gives a better result. The statistical quantities become

mean value \bar{y}	0.16	(0.26)
standard deviation σ_y	0.40	(0.63)
correlation $r_{x_1 y_{quad}}$	0.46	(0.63)
correlation $r_{x_2 y_{lin}}$	0.86	(0.77)

and 99.7% of the samples will have a function value less than 1.36.

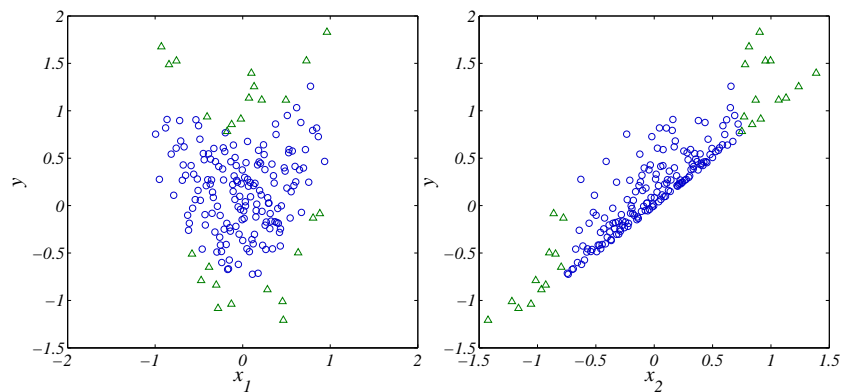


Figure 6.14: Scatter plot of y and the input variables for input variable combination 3. All samples with x_1 smaller than -1 ($\bar{x}_1 - 3 \cdot 0.33$) or larger than 1 ($\bar{x}_1 + 3 \cdot 0.33$) are removed, samples with x_2 smaller than -0.75 or larger than 0.75 are marked with triangles.

To alter the variation of the variables through a removal of samples outside of the limits is of course an approximation. If the robustness analysis is executed once more, now with $\sigma_{x_1} = 0.33$ and $\sigma_{x_2} = 0.25$, the following statistical quantities are achieved

mean value \bar{y}	0.10	(0.26)
standard deviation σ_y	0.28	(0.63)
correlation $r_{x_1 y_{quad}}$	0.49	(0.63)
correlation $r_{x_2 y_{lin}}$	0.88	(0.77)

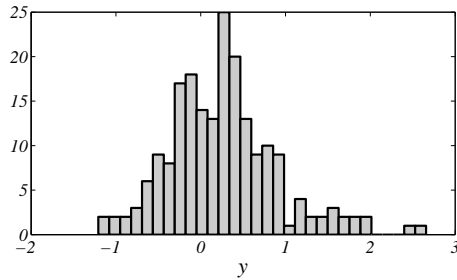


Figure 6.15: Histogram of y with input combination 3, x_1 and x_2 normally distributed, $\sigma = 0.5$.

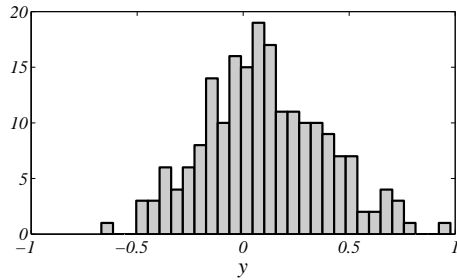


Figure 6.16: Histogram of y with reduced variances of the input variables, x_1 and x_2 normally distributed, $\sigma = 0.33$ and $\sigma = 0.25$ respectively.

That gives an upper 99.7%-limit of 0.94 ($\bar{y} + 3 \cdot \sigma_x$). Histogram of y with the original distributions and a histogram of y with the improved distributions are presented in figures 6.15 and 6.16 respectively.

In the case of large FE-models it is mostly not possible to redo the robustness analysis with the improvement and the approximation must be enough. In many practical examples, it is easier to reduce the variance of some variables than others. This must be considered in the proposal of an improvement strategy. In some cases it is necessary or perhaps easier to change the design itself (in this case the function) to achieve robustness.

6.2 Robustness evaluation of a foam model

6.2.1 Objectives

When simulating an entire car it is important to model the materials properly. Especially difficult due to more complex behaviour and less available research than for metals, are the plastics and for example the foam in the seats. In these cases it is important to find a material model that fulfils all requirements without bringing extra demands on computing effort in the simulation. Often, real experiments are performed and set as reference and the computer model are fitted to the experimental data.

In this investigation an experiment has previously been performed with a cube of the foam material used in a car seat. On the top and the bottom of the foam cube, metal plates were placed with contact to the foam. The cube was compressed with a constant, high velocity to 10% of its original height. The foam's transverse contraction is zero which implies that the cross-section area is constant during the entire compression. The applied force was measured at a point in the middle of the lower metal plate. The stress is derived by dividing the applied force by the cross-section area of the cube (constant through the compression). The experiment was performed with four different strain rates and the collected stress-strain curves are used in the simulation both as material parameters in the FE-model and in the verification of the simulated curves.

The objective of the original investigation was to find a material model for a FE-analysis that show the same characteristics as the model in the experiment. A FE-model with the same structure and size as the experimental model was created. The parameters in the constitutive model was fitted to the experimental stress-strain curves.

During this initial investigation it was noticed that the stress-strain curve obtained in the simulation sometimes oscillates around the experimental curve used for the validation of the material card, see figure 6.17. A possible reason is that a wave in the material arises when a foam material is being compressed with high velocity in the simulation. The wave travels back and forth through the cube and influences the stress measurements which causes the oscillation. The phenomenon is equal to a sound wave that travels through the air.

The objectives of the robustness evaluation is to determine which parameters in the simulation affect the magnitude of this oscillation.

6.2.2 The FE-model

The foam model is a simple cube with solid elements. Friction between the metal plates and the foam is defined to improve the stability of the model. The metal plates are pressed against each other with constant velocity until the foam is compressed to

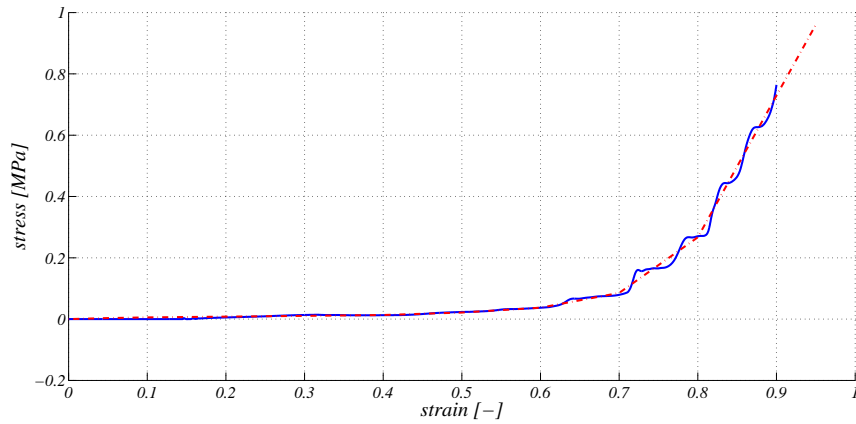


Figure 6.17: The stress-strain curve obtained with measurements in an experiment used for validation (dashed) and the stress-strain curve after a FE-simulation of the foam material (solid) at strain rate 57.33 s^{-1} . The simulated curve is an example of a large oscillation.

10% of its original thickness. The compression is proceeded in 0.0157 s which means a strain rate of 57.33 s^{-1} .

The material is an, in ABAQUS, build-in user material called *Rate Sensitive Foam* with parameters especially defined for crash analysis. The transverse contraction of the material is zero. The stress-strain relationship of the material is depending on the strain rate of the element and is obtained in the previously explained experiments. Different strain rates give different relationships, see figure 6.19.

The result of the simulation is the applied force over the time of the compression. This curve is converted to a stress-strain curve. The stress is the *Cauchy stress*, the force divided by the cross-section area of the deformed model (equal to the initial area

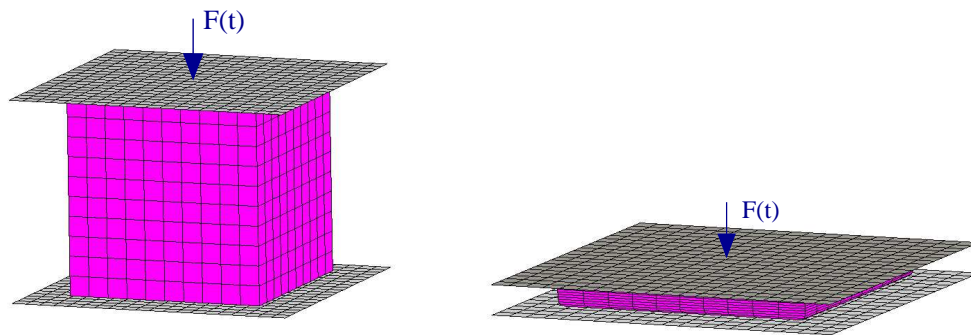


Figure 6.18: The undeformed and deformed foam model with the FE-mesh visible.

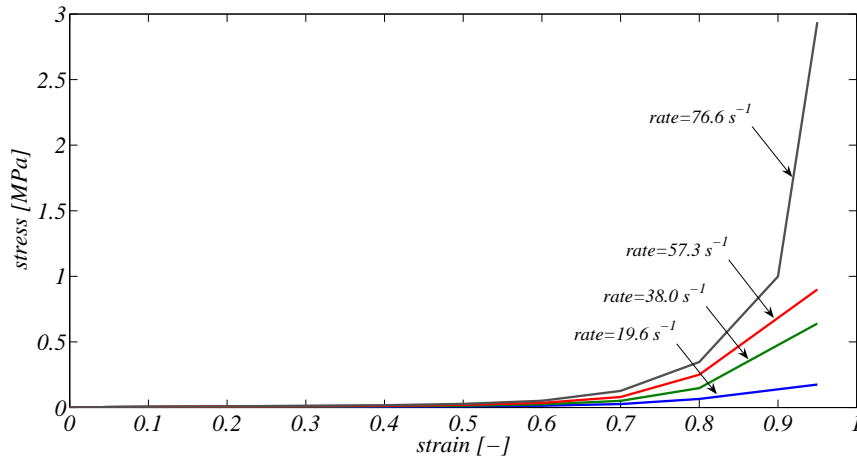


Figure 6.19: The stress-strain curves of the material for 4 different strain rates. The curve with strain rate 57.33 s^{-1} is the curve used as verification in the robustness analysis.

since the transverse contraction is zero). The strain is defined as $\int_T \dot{\delta}/L dt$ where L is the original height of the cube. The compression is defined to proceed with constant velocity, *strain rate*, so the strain is obtained by multiplying the time with the strain rate.

6.2.3 The robustness analysis

3 material parameters and 2 parameters for the simulation settings are considered as uncertain parameters in the model. The material parameters are the density of the foam, the friction between the metal plates and the foam cube and an exponent that controls the hardening in the stress-strain curves. The curves are varied by percentage variation of the stresses at constant strains, see figure 6.20. Each curve (for each strain rate) is varied with the same factor. In the simulation settings the length ratio for the distortion control and the quadratic bulk viscosity are varied. The distortion control prevents the elements from being distorted when compressed more than the defined length ratio. The bulk viscosity is a damping parameter associated with volumetric strain [ABAQUS]. The distributions for the different uncertain parameters are displayed in table 6.6.

The maximum distance between the simulated and the experimental stress-strain curve and the area between the two curves are to be investigated as system responses. The strain interval is divided into 4 parts, 0 to 0.25, 0.25 to 0.5, 0.5 to 0.75 and 0.75 to 0.9, see figure 6.21. The maximum difference and area between the curves are calculated in each interval, i.e. a total of 8 response functions are investigated, see table 6.7.

Parameter	Distribution
Density	normal, $\sigma_x = 0.1 \cdot \bar{x}$
Friction	normal, $\sigma_x = 1/3 \cdot \bar{x}$
Stress-strain curve	normal, $\sigma_x = 0.1 \cdot \bar{x}$
Distortion control, length ratio	uniform, $x = [0, 0.3]$
Quadratic bulk viscosity parameter	normal, $\sigma_x = 1/6 \cdot \bar{x}$

Table 6.6: The distributions of the uncertain parameters.

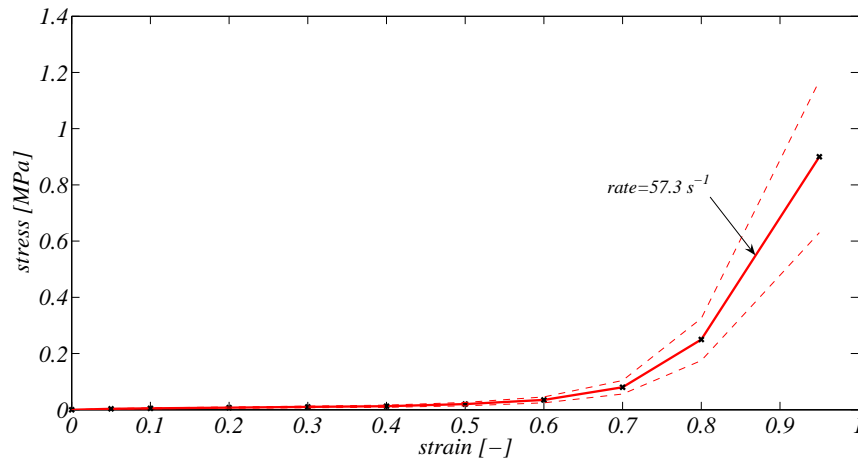


Figure 6.20: The variation of the stress-strain curve is made by a percentage increase/decrease stress at 10 strain values. The dashed lines show a variation of the curve with $\bar{x} \pm 3 \cdot \sigma$ i.e. a 30% increase and decrease of the stress at the corresponding strains.

The robustness analysis is performed with Latin Hypercube Sampling with 200 samples in DAKOTA [3]. Each sample is simulated in ABAQUS/Explicit [1] and the simulated curve is compared to the experimental in METApost [4]. The evaluation of the sample data is made in the toolbox developed within this thesis and described in section 5.1.

6.2.4 Result

The first results of the response functions in the robustness analysis of the foam model are displayed in table 6.8. The objective of the robustness analysis is to find the cause of the oscillation of the stress-strain curve. The most important quantity is the coefficients of correlation. The density and/or the strain-curve are the only two input

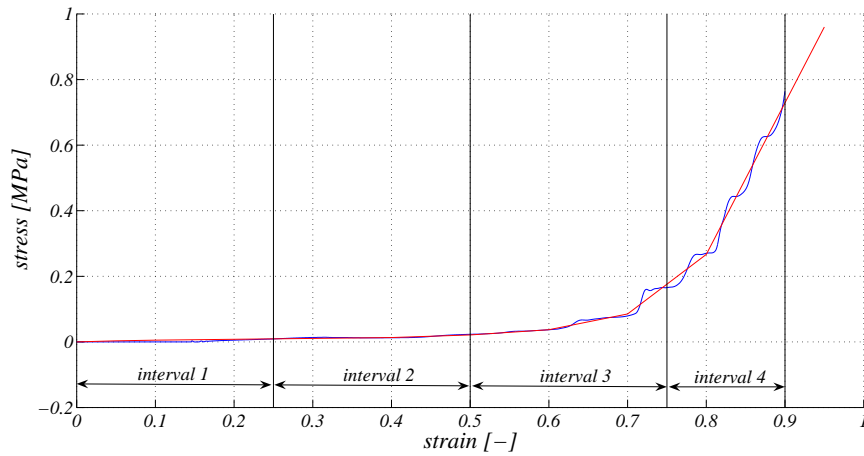


Figure 6.21: The maximum distance and the area between the simulated and the experimental stress-strain curve are evaluated in 4 intervals of the strain.

Investigated system responses

- dist1** Maximum distance between the simulated and the experimental curves used for validation in the first time interval i.e. the first 25% of the compression.
- area1** Area between the two curves in the first interval.
- dist2** Maximum distance between the curves in the second interval.
- area2** Area between the two curves in the second interval.
- dist3** Maximum distance between the curves in the third interval.
- area3** Area between the two curves in the third interval.
- dist4** Maximum distance between the curves in the last interval.
- area4** Area between the two curves in the last interval.

Table 6.7: The system responses investigated for robustness.

variables with important correlation to the responses after a linear regression model. The first interval has very high correlation coefficients. It is clear that the density influences the area between the simulated and the experimental stress-strain curves and the location of the experimental curve influences the maximum distance between the two curves.

The difference between the simulated stress-strain curve and the experimental curve used for verification at the beginning of the compression, see figure 6.22, is caused by a form of inertia of the model. The cube has already been slightly compressed before any stress is detected at the measurement point at the bottom of the cube. This inertia

	mean [MPa]	CV	R_{lin}^2	important correlations
dist1	0.01	7.9%	99.5%	stress-strain curve ($r = 0.94$)
area1	7.4e-4	7.7%	100%	density ($r = 0.91$) and stress-strain curve ($r = 0.40$)
dist2	2.5e-3	10.2%	78.4%	density ($r = 0.88$)
area2	2.8e-4	15.8%	82.8%	density ($r = 0.89$)
dist3	0.02	16.3%	88.3%	density ($r = 0.89$)
area3	8.3e-4	13.6%	69.4%	density ($r = 0.81$)
dist4	0.04	32.3%	47.1%	density ($r = 0.36$) and stress-strain curve ($r = 0.45$)
area4	2.6e-3	17.0%	39.2%	density ($r = 0.52$)

Table 6.8: The mean values, coefficient of variance CV , sum of coefficients of determination R_{lin}^2 and important linear correlations of the eight response functions.

increases with an increased density.

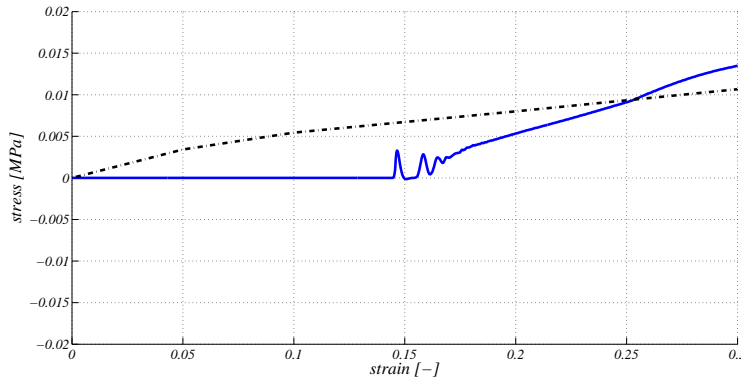


Figure 6.22: The inertia of the model causes a delay in the stress reaction in the simulated model. The simulated curve is solid and the curve obtained in the experiments is dashed.

The variation of a response is well explained by the correlation coefficients with the chosen regression model if the *sum of coefficients of determination* R^2 for this response is close to 100%. The five first responses have relatively high values of R^2 and the variables with the highest correlation coefficient, i.e. the *density* and the *stress-strain curve* are the variables with the strongest influence on each response respectively. This is the first conclusion of the robustness analysis.

Only 47.1% and 39.2% of the variance of the responses $dist4$ and $area4$ respectively

are explained with the linear correlation analysis. The variance of the response $area_3$ is with 69.4% also relatively poorly explained. The reason for the unexplained variance of these three responses must be found.

One reason for this unexplained variation was found when investigating the scatter plots between the input variables and the response $dist_4$. The response $dist_4$ is distinctly increased for some samples with a high value of the *distortion control length ratio*, see figure 6.23. This indicates that the simulation itself is unrobust. A value of the *distortion control length ratio* above 0.07 should be avoided for a more robust simulation with more reliable results. This is the second conclusion of the robustness analysis. Instead of removing all samples with a value of the *distortion control length ratio* above 0.07 only the samples where it is clearly seen that the distortion control have interfered are removed from the rest of the robustness evaluation.

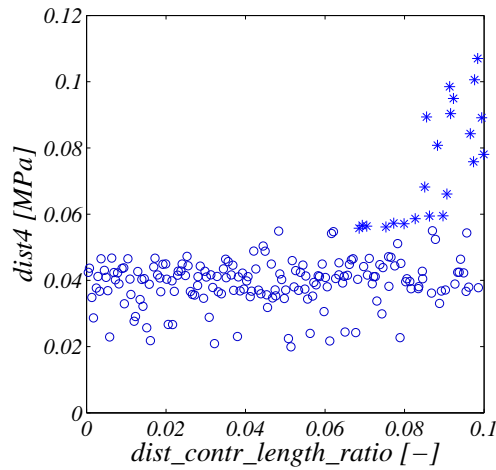


Figure 6.23: The samples where the distortion control have interfered and increased the stress (*) and the samples where the distortion control did not interfere (o).

The removal of the samples caused changes in the statistics in the very last strain interval, see figure 6.24. The new values of the quantities for the three remaining responses are displayed in table 6.9. The distortion control interferes when the elements are highly compressed, i.e. only in the last strain interval. The influence of the removal of the diverging samples is negligible in the other three intervals.

The variance of the response $dist_4$ has now been reduced. The sums of coefficients of determination show that the variances of $dist_4$ and $area_4$ are even less explained by the linear correlation analysis than before. Additional explanations must be found.

Relatively few variables are involved in this robustness analysis. This means a relatively few number of combinations to be plotted and it is easy to go through all scatter plots.

	mean [MPa]	CV	R_{lin}^2	important correlations
area3	8.3e-4	13.7%	70.3%	density ($r = 0.80$)
dist4	0.04	17.4%	29.0%	stress-strain curve ($r = 0.46$) and density ($r = 0.36$)
area4	2.5e-3	16.9%	34.7%	density ($r = 0.46$)

Table 6.9: The mean values, coefficient of variance CV , sum of coefficients of determination R_{lin}^2 and important linear correlations of the three remaining response functions.

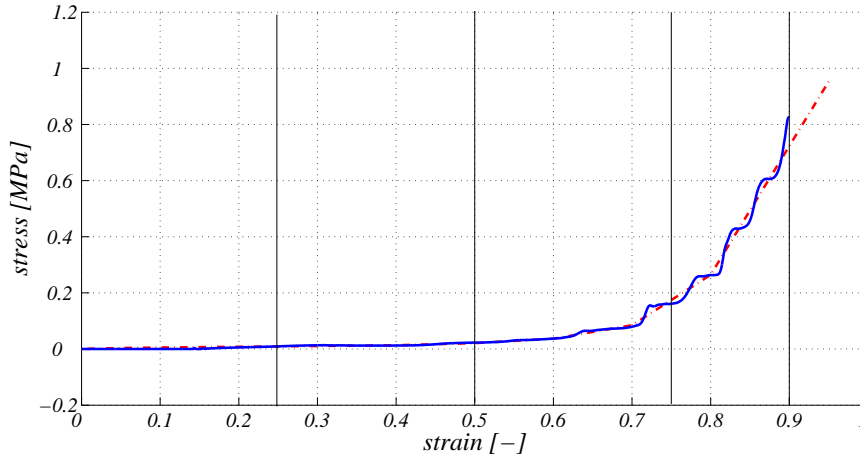


Figure 6.24: The interference of the distortion control causes an increase of the stress in the very last part of the compression.

A quick investigation of all scatter plots does not give any hints that a regression analysis of higher degree will bring new information. Some patterns of the samples can however be identified in the plots with the input variables *density* and *stress-strain curve* and the response *area4*. By investigating the location of individual samples in the different scatter plots parallel it is possible to identify more complex relationships. In figure 6.25 four interesting scatter plots are displayed. One group of samples are marked with squares and another group is marked with triangles. The area in the third interval is normally correlated to the area in the fourth interval, i.e. the oscillations are in most cases equally large in interval 3 and 4. The samples marked with triangles have smaller areas than expected in interval 4 but relatively large areas in interval 3, i.e. they oscillate more in interval 3 than in interval 4, see figure 6.27. The samples marked with squares have small area in both intervals, the area in interval 4 is much smaller than expected by the linear regression model. The simulated stress-strain curve and its validation curve is displayed in figure 6.28.

The two groups can be identified as being combinations of high values of *density* with low values of *stress-strain curve* or low values of *density* with high values of *stress-strain curve*, see figure 6.26. Responsible for this variation is a combination of the *density* and *stress-strain curve*. The reason for the unexpected large and small responses might be a form of eigenfrequency of the material. At some combinations of density and strain-curve and at some points of the compression, the wave that arises in the compression is amplified or reduced. This causes increased or decreased oscillations of the curve. Exactly when this change occurs is very hard to foresee and depends on the strain rate. This is the third conclusion of the robustness analysis.

After the removal of the samples suspected to be influenced by some form of eigenfrequency, the correlation analysis is repeated. The new statistics for the remaining responses are displayed in table 6.10.

	mean [MPa]	CV	R_{lin}^2	important correlations
area3	7.8e-4	10.14%	82.8%	density ($r = 0.81$)
dist4	0.04	11.4%	59.0%	density ($r = 0.56$) and
area4	2.5e-3	16.9%	100%	density ($r = 0.81$)

Table 6.10: *The mean values, coefficient of variance CV, sum of coefficients of determination R_{lin}^2 and significant linear correlations of the response functions remaining in the evaluation.*

The rest of the variances of the responses *area3* and *area4* are completely explained by the linear correlation analysis and the density is in both cases the most influent variable. Some of the variance of *dist4* remains unexplained. It is difficult to identify the factor responsible for this last variation. The most probable reason is that the high deformation of the elements in the last part of the compression can lead to numerical problems in the simulation with variations in the response as result. This is the final conclusion of the robustness analysis.

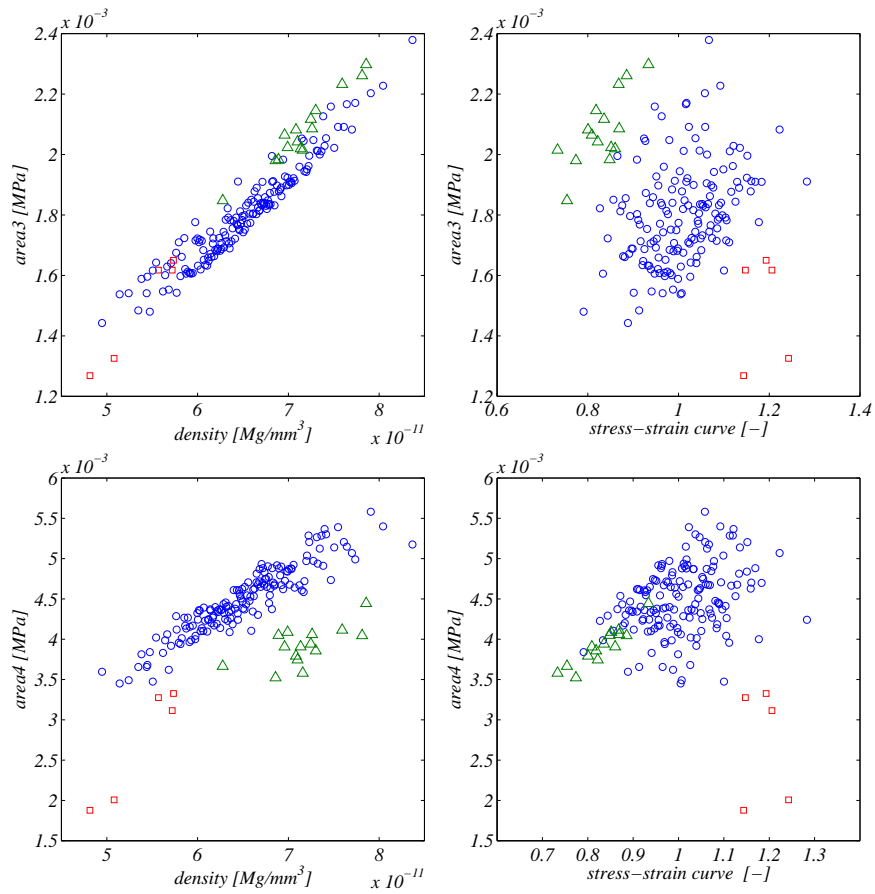


Figure 6.25: Two groups of samples deviate slightly from the others in the plots with the input variables density and stress-strain curve and the response area4. They are marked with squares and triangles respectively.

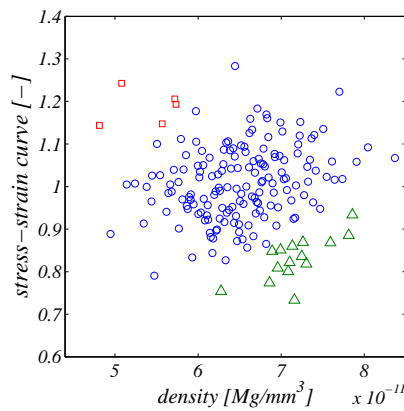


Figure 6.26: The two groups of samples in figure 6.25 seems to have a certain relationship between density and stress-strain curve.

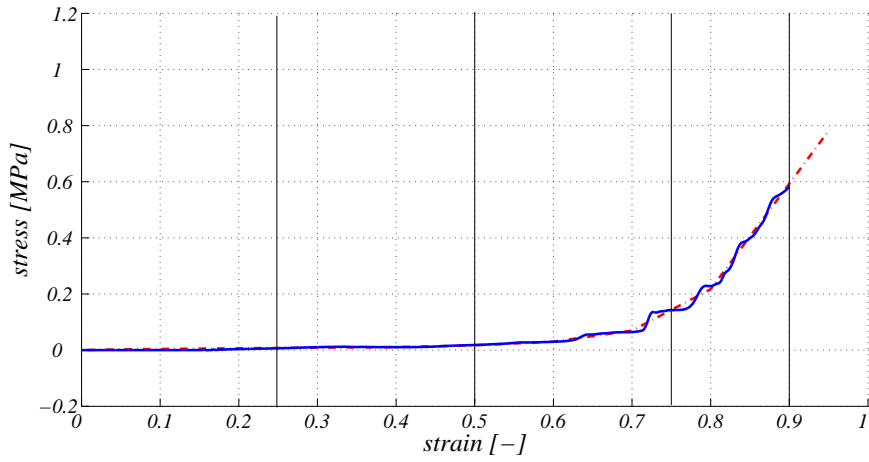


Figure 6.27: A typical stress-strain curve for one of the triangular samples. The curve oscillates a lot between a strain of 0.7 and 0.8 and less in the last part of the compression.

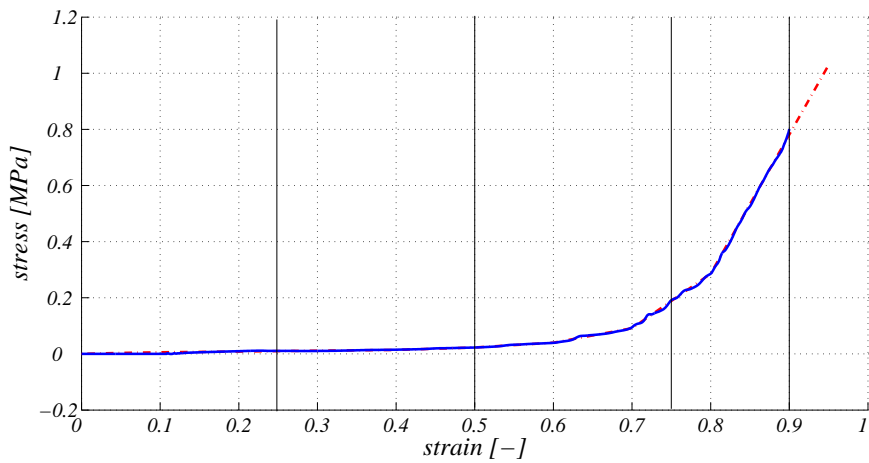


Figure 6.28: A typical stress-strain curve of one of the square samples, the curve oscillation is extremely small.

6.2.5 Conclusions

Four conclusions could be drawn from the robustness evaluation of the foam model.

- The value of the *distortion control length ratio* shall be less than 0.07 for a robust simulation.
- The *density* and the *stress-strain curve* are the only influent variables in the first half of the compression.
- The *density* is the most influent variable in the second half of the compression. A few samples with special combinations of *density* and *stress-strain curve* cause unexpected amplification or reduction of the oscillation.
- Some variations in the last part of the compression are difficult to explain. One probable reason is numerical problems due to highly deformed elements.

6.3 Robustness analysis of a acoustic simulation of a rear floor panel in a car body.

6.3.1 Objectives

The robustness of the acoustic performance of a rear floor panel of a car body, see figure 6.29, is to be analysed. The variance of the acoustic performance of the model due to small variations in the model geometry is investigated. The acoustic performance is evaluated in form of the *Equivalent Radiated Power* (ERP) over a defined frequency interval. The ERP is defined as

$$ERP = 1/2 \cdot \rho_F \cdot c_F \cdot \int_A v_n^2 dA \quad (6.4)$$

where ρ_F is the density of the fluid (air in this case), c_F is the sound velocity in the fluid, v_n is the normal velocity of the radiating surface and A is the area of the radiating surface.

The part have previously been optimised with respect to the ERP. Beads have been inserted in a specific pattern on the panel to reduce the radiated power over a frequency interval. The optimisation process of the part has resulted in a geometry that shows improved acoustic performance [REGER *et al.* 2006]. It is desired to investigate how sensitive this improved acoustic performance is to small variations in the geometry of the part i.e. if it is possible to manufacture the part with unchanged performance. The variation of the acoustic performance due to uncertainties must not be larger than the improvements achieved by the optimisation.

The objective of the robustness analysis is to answer the questions:

- How does the ERP vary in the defined frequency interval due to small variations in the geometry.
- Which areas of the optimised design are especially sensitive for geometry variations regarding the variation of the Equivalent Radiated Power of the entire part?

6.3.2 The FE-model

The geometry of the optimised rear floor panel is displayed in figure 6.30. The car is excited with a time-dependent force from the engine.

The parametrisation of the FE-model is made with *morphing* in ANSA [2]. The use of morphing is an alternative to an individual parametrisation of each FE-node. The FE-model is divided into a number of *control boxes* with *control points* in each corner

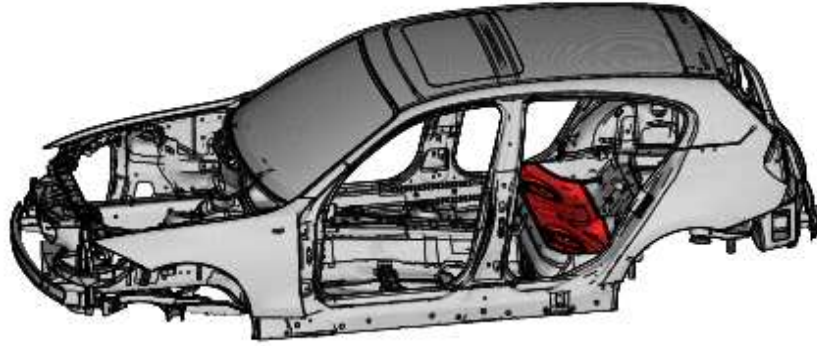


Figure 6.29: The part to be investigated for robustness is marked red.
(Courtesy of BMW)

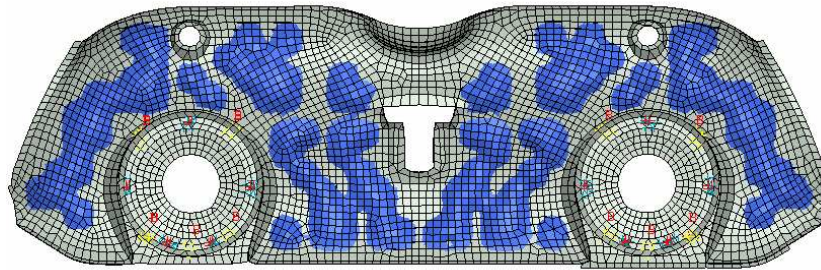


Figure 6.30: The optimised rear floor panel, the blue areas correspond to the inserted beads.

that control the location of the nodes in the FE-mesh. The control points can easily be parameterised and by that arbitrary geometry variations of the FE-model can be obtained. The result of moving one control point normal to the surface is displayed in figure 6.31. The coarse mesh show four control boxes with corresponding control points that are used in the parametrisation. The finer mesh corresponds to the FE-mesh.

The optimisation was made with a discrete variation of the control points. The control points were either displaced 10 mm away from the surface or left in its original position. The optimisation resulted in a pattern of beads with displaced control points, 52 points were displaced and the remaining 56 points were left in its original place. This optimised geometry is to be evaluated for robustness. Figure 6.32 show the location of the control points on the floor panel and where the beads are placed. More information about the model and the optimisation process can be found in [REGER *et al.* 2006].

The main FE-analysis is performed with Nastran [5]. A frequency response analysis is a relatively expensive analysis and different actions can be taken in order to reduce

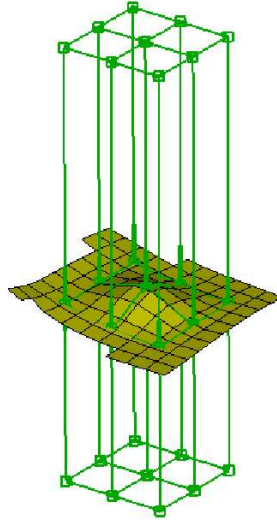


Figure 6.31: A modified FE-mesh by the use of morphing. The central control point is moved normal to the surface.

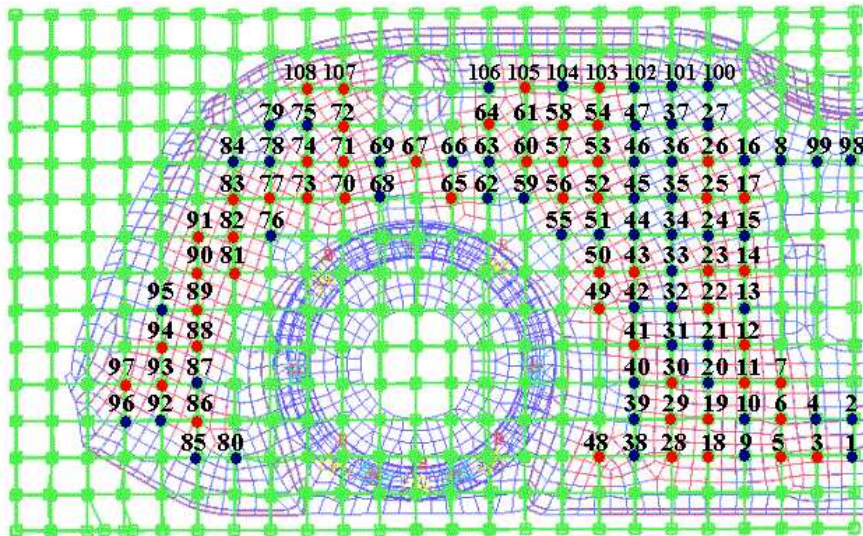


Figure 6.32: The 108 morphing control points, p_i , used as input variables in the robustness analysis. The part is symmetrical parametrised. The red points have a mean value of -10 mm (bead), the blue points have a mean value of 0 mm (no bead).

the computing time. The dynamic behaviour of the rear floor panel is affected by the surrounding environment and the entire car must be included in the analysis. This leads to a calculation time of several hours. The computing time can be considerably reduced by the use of *substructuring*, for example the *superelement technique* in Nastran, [REGER *et al.* 2006]. Another action to reduce the computing time is the use of ACMS (*Automated Component Mode Synchronisation*). ACMS automatically divides a large problem into smaller problems via domain decomposition and solves each of the smaller problems individually. ACMS is designed to be used on very large-scale dynamic analyses with the objective of reducing the computing time.

6.3.3 The robustness analysis

The robustness analysis is made with Latin Hypercube Sampling with a sample size of 200. The iteration loop in this example is divided in several steps and each step uses a specific software. The sample generation is made in DAKOTA [3] which also controls the iteration loop. ANSA [2] uses the variable combinations from DAKOTA to build the geometry using *morphing*. The output from ANSA is transferred to Nastran [5] where the *frequency response analysis* is performed. The ERP of the entire model for different frequencies is derived in Sysnoise [6] and returned to DAKOTA as result.

The locations of all morphing control points p_i used in the optimisation process are defined as uncertain geometry points in the robustness analysis, see figure 6.32. They are assumed to be normally distributed with means in the design optimum (0 or -10 *mm*) and varied in z -direction with a standard deviation of 0.5 *mm*. A total of 108 geometry points are considered uncertain and varied with the use of *morphing*.

The ERP in 281 frequencies, logarithmic distributed between 25 *Hz* and 400 *Hz*, are calculated. Additional response functions are the maximum ERP in the frequency interval, the mean of the 50 highest values of the ERP in the interval and the overall ERP mean. Altogether 284 response functions are evaluated.

6.3.4 Results with ACMS

The results of the calculations are shown in figure 6.33. Each line represents the ERP over the frequency interval of one sample. It can clearly be seen how the ERP varies between the samples.

In this problem a large variance of the ERP affects the acoustic performance more for a high mean value than for a low. The *coefficient of variance*, defined as the standard deviation divided by the mean value, is for that reason a bad measurement. A response with a high mean will have a lower coefficient of variance than a response with a low mean if the variance is the same. The measure of variance to be investigated in this problem is therefore the *standard deviation*. The mean value of the ERPs over the

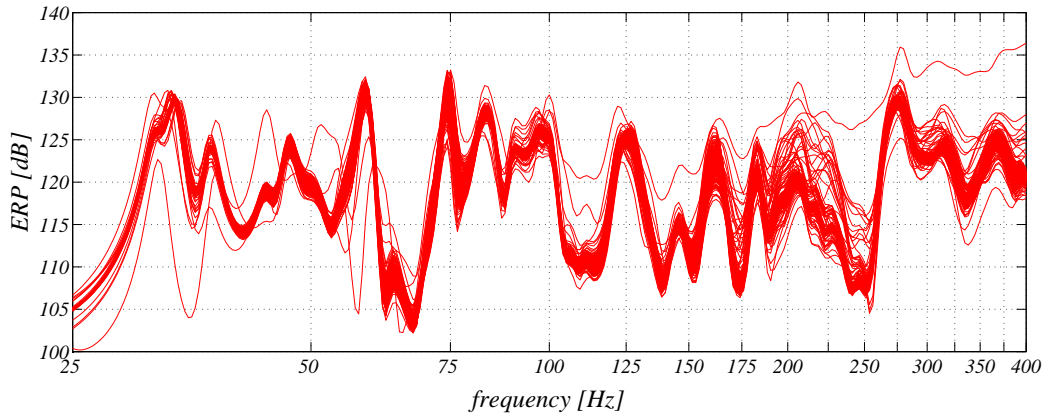


Figure 6.33: The equivalent radiated power, ERP, of the 200 samples obtained in the robustness analysis.

frequency interval varies between 103 dB and 130 dB and the standard deviations varies between 0.4 and 2.7, see the histograms in figure 6.34.

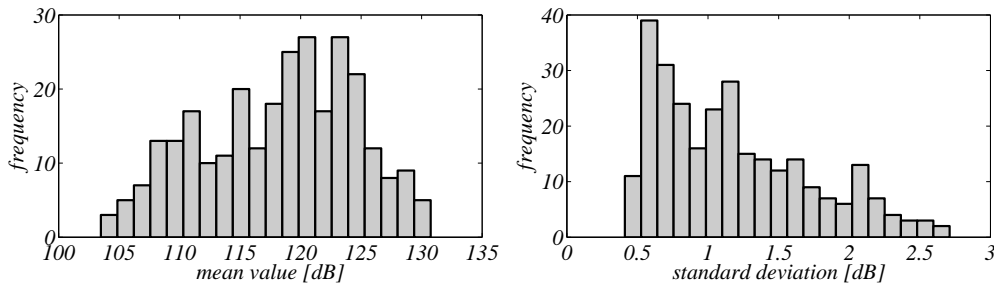


Figure 6.34: Histograms of the mean value and the standard deviation of the ERPs at the different frequency levels.

It is of interest to analyse the result more thoroughly in order to locate the origin of the variation. Initially a linear correlation analysis of the input variables with respect to the response functions is made. The corresponding sums of coefficients of determination R^2 , see section 2.2.4, for the response functions at the different frequency levels are displayed in figure 6.35. The plot shows that only about 60% of the variance is in some frequency intervals explained with a linear regression analysis. It is necessary to try other evaluation methods to find the reason for the remaining variation.

A quadratic regression analysis with a following correlation analysis is made. A quadratic regression analysis can be seen as a multiple linear regression analysis where new variables, x_i^2 , are introduced. The input variables x_i and x_i^2 are correlated to each other if the mean value of x , \bar{x}_i , is significantly different from zero. In this case, about half of the input variables have a mean value of -10 mm and x_i and x_i^2 are strongly correlated. This means that we no longer can calculate R^2 with the sum of the squared

correlation coefficients. A way to do this is briefly explained in section 2.2.4. This is a rather complicated procedure and will not be performed here. Instead we compare the achieved quadratic correlation coefficients with the linear. No “new” correlations are detected and we assume that the variation of ERP unexplained after the linear regression analysis remains unexplained after the quadratic regression analysis.

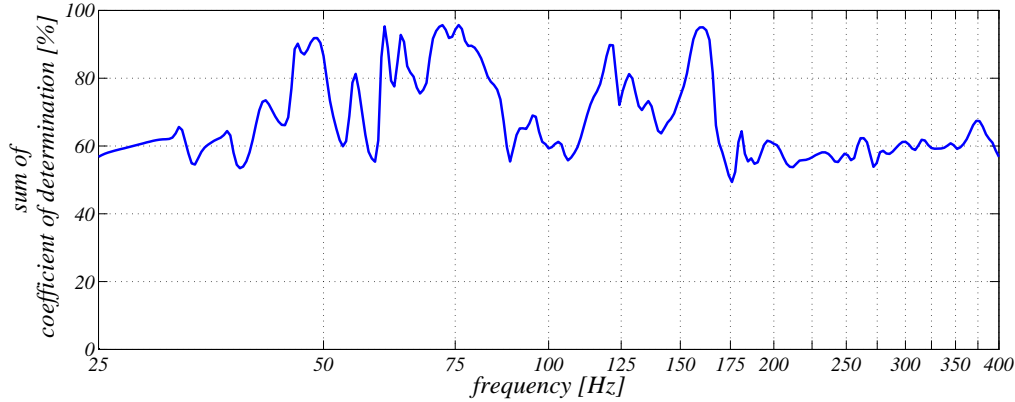


Figure 6.35: *The sums of the coefficient of determination of the response functions calculated with ACMS (the ERPs at the different frequency levels) after a linear correlation analysis.*

An analysis of the scatter plots also does not bring additional information about the variation. This means that the variation of the ERP comes from an uncertain variable that is not included as input variable in the analysis. In this case it is suspected that the numeric approximation method ACMS used in the frequency response analysis is the reason for the remaining variation of the response. To see how ACMS influences the result, the entire robustness analysis is remade without ACMS. All other settings are unchanged and samples with the same values of the input-variables are generated.

6.3.5 Result without ACMS

The results from the calculations without ACMS differs distinctly from the calculations with ACMS in some frequency intervals. Especially at frequencies above 175 Hz the variation of the samples calculated with ACMS is much larger than of the samples calculated without ACMS. The assumption that ACMS is responsible for some of the variance in the previous analysis seems to be correct. The difference between the responses in the two robustness analyses is displayed in figure 6.36.

Nine of the samples calculated without ACMS have significantly different response values than the rest in large parts of the frequency interval. They are marked in figure 6.37. The result of the frequency response analysis, i.e. the nodal displacements, of these samples are visually investigated to see if the calculations are correct. Discontinuities in the displacements in three of the samples indicate that the frequency

response analysis have failed in these cases. A closer analysis of the input data of the samples does not indicate anything unusual and it is assumed that the difference of the response values does not originate from the variance of the geometry. These three samples are removed from the analysis. The reason for the failure is unknown and might be investigated later. It is more difficult to find indicators of failures in the calculation in the rest of the samples with deviating response values. They are therefore kept in the evaluation.

A linear regression analysis with a following correlation analysis of the population is made. The sums of the coefficients of determination are displayed in figure 6.38. When comparing the graph of the sums of the coefficients of determination and the graph with the ERP it can be noticed that the sums of the coefficients of determination are lower in the areas where the six deviating curves are far from the rest of the curves. This indicates that the variance in this areas, i.e. mainly the deviation of the six marked curves, is not caused by the input variables but from something else. It is of course very important to find the reason for this divergence for the evaluation of robustness for the entire simulation. The main objective of this robustness analysis is however to find sensitive areas in the geometry and the six samples are removed from the remaining evaluation.

The linear correlation analysis is made once more and this time the sums of the coefficients of determination are close to 100% in the entire frequency interval, see figure 6.39. All variation is explained with the linear correlation coefficients.

The ERPs for the samples left in the analysis are displayed in figure 6.40. To simplify the evaluation the number of response functions to be evaluated is reduced. Five especially interesting frequencies where both the mean value and the variance are large are located.

The responses of interest are reduced to:

1. The ERP at 58.6 *Hz*.
2. The ERP at 74.3 *Hz*.
3. The ERP at 125.6 *Hz*.
4. The ERP at 162.5 *Hz*.
5. The ERP at 369.5 *Hz*.
6. The maximum ERP over the entire frequency interval.
7. The mean value of the highest 50 ERPs.
8. The mean value of the ERP over the entire frequency interval.

Mean values, standard deviations and sum of coefficients of determination of the ERPs

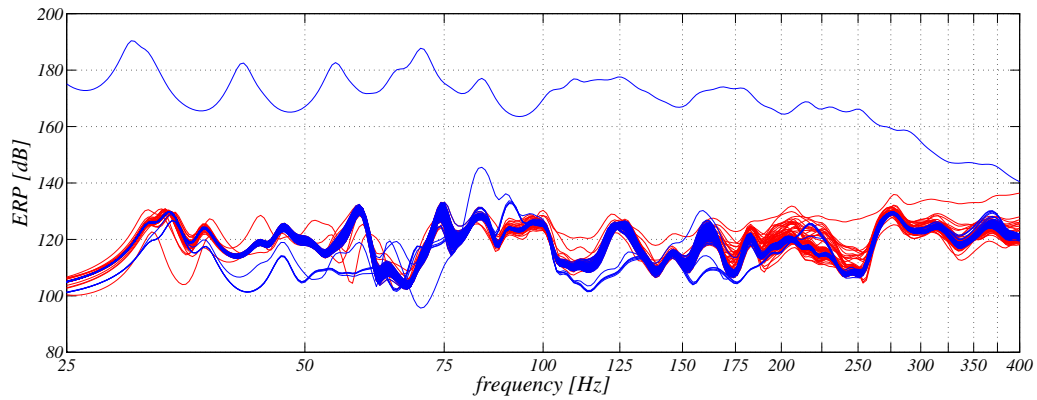


Figure 6.36: The equivalent radiated power, ERP , of the samples obtained with ACMS (red) and without ACMS (blue) in the robustness analyses.

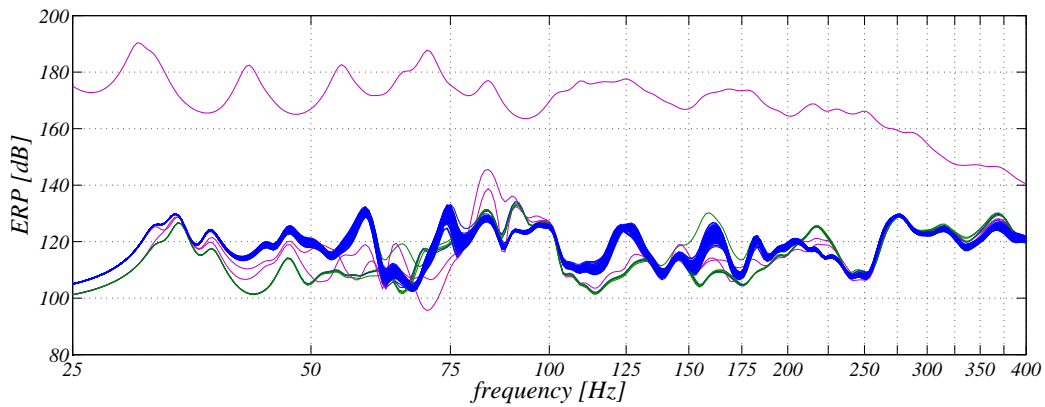


Figure 6.37: The equivalent radiated power, ERP , of the samples obtained without ACMS in the robustness analysis. The samples with magenta curves have not been calculated correctly and are removed from the analysis. The green curves also differ significantly from the others but the reason is unknown.

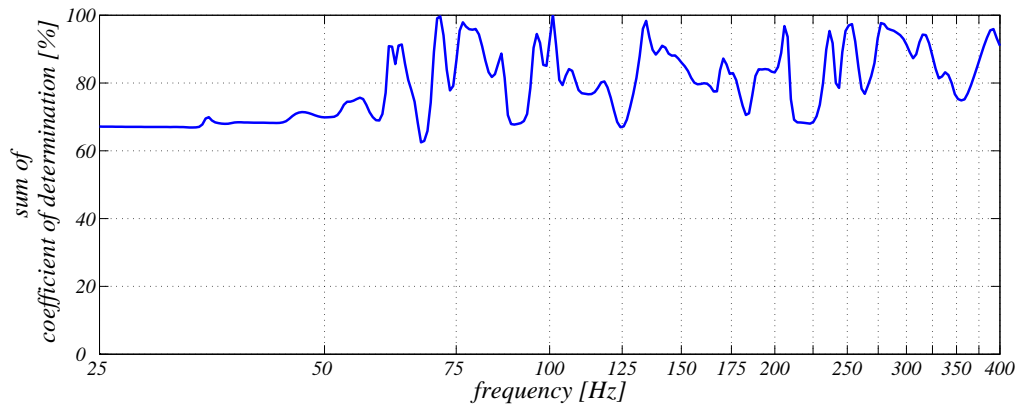


Figure 6.38: The sums of the coefficient of determination of the response functions. The green curves in figure 6.37 are still included but the magenta curves have been removed.

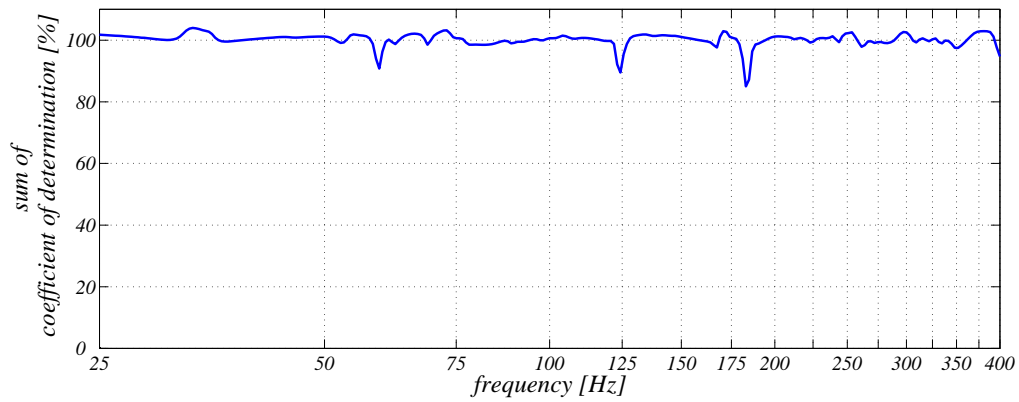


Figure 6.39: The sums of the coefficient of determination of the response functions. The green and the magenta curves in figure 6.37 have been removed.

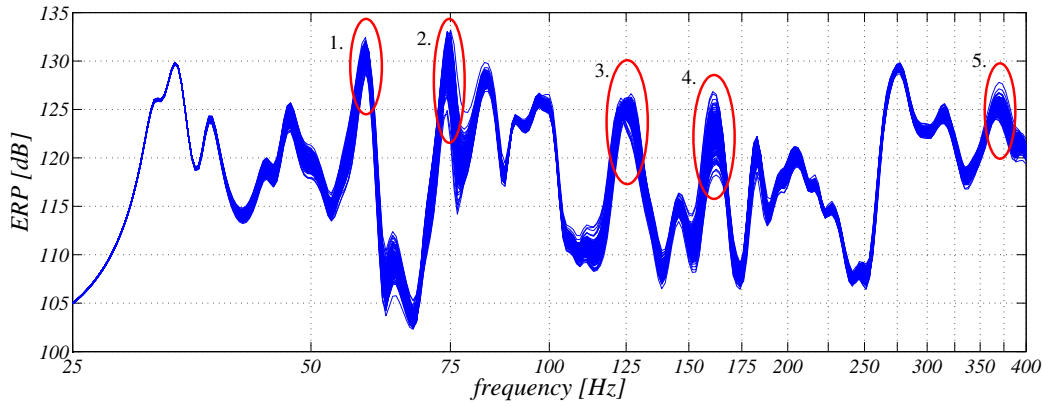


Figure 6.40: The equivalent radiated power, ERP , of the remaining 191 samples obtained in the robustness analysis without the approximation method $ACMS$.

and input variables with important correlation coefficients with respect to each ERP respectively are displayed in table 6.11. The varied geometry points, p_i , are the points displayed in figure 6.32 on page 84.

	mean	σ_{ERP}	R^2	important linear correlations
1.	129.4	0.84	98.8%	p_{44} ($r=0.54$), p_{51} ($r=0.35$) and p_{46} ($r=0.3$)
2.	130.0	1.51	101%	p_{37} ($r=-0.30$) and p_{99} ($r=-0.30$)
3.	125.0	0.65	95.5%	p_{34} ($r=0.49$)
4.	122.5	1.62	99.6%	p_{37} ($r=0.49$), p_8 ($r=0.4$) p_{44} ($r=-0.34$) and p_{33} ($r=-0.25$)
5.	125.3	0.61	102%	p_{35} ($r=0.26$), p_{47} ($r=0.26$) p_{85} ($r=-0.26$) and p_{58} ($r=-0.25$)
6.	131.0	0.76	93.0%	p_{44} ($r=0.40$), p_{51} ($r=0.34$) and p_{98} ($r=-0.29$)
7.	126.8	0.24	101%	p_{37} ($r=-0.31$), p_{51} ($r=0.27$)
8.	118.1	0.13	102%	p_{34} ($r=-0.26$), p_{51} ($r=0.28$) p_{98} ($r=-0.29$) and p_{99} ($r=-0.29$)

Table 6.11: The mean values, standard deviation, sum of coefficients of determination and morphing control points, p_i , with important linear correlations to the eight most interesting response functions.

6.3.6 Conclusion

The first result from the robustness analysis is that the ACMS feature in Nastran is not suitable for this problem. ACMS increased the variation of the response in this model, in this case the ERP, by 60%. This is of course not tolerable. Performing the calculation without ACMS improved the variation in large frequency areas but the computing time increases significantly. The iteration loop needs approximately one hour with ACMS and up to six hours with the normal solver. This is a big problem and it is perhaps not possible to integrate robustness analyses of acoustic-problems in the product development process due to the long computing times without ACMS.

Surprising is that a variation due to numerical uncertainties exist even without ACMS. 9 of the 200 samples calculated without ACMS had a divergence from the other samples that can not be directly explained by the variation of the model geometry. This variation is too large to consider the simulation robust. The reason for this divergence is still unknown.

After removing the diverging samples, the influence of the small variations of the geometry on the ERP can be investigated. The robustness of the model geometry is analysed. The variances of the overall mean value of the ERP and the mean value of the 50 highest values of the ERP are small, see table 6.11. More interesting is to locate the cause of the variance of single peaks in the ERP curve and try to improve them. It was possible to locate five especially critical frequencies where both the mean value and the variation of the ERP are large. The variation of the ERP at these frequencies is fully explained by a linear correlation analysis (the sum of coefficients of determination is close to 100%).

The obtained levels of correlation coefficients are rather small in all cases (≤ 0.6). This, together with the high sums of the coefficients of determination, indicate that several geometry points have equal influence on the ERP. A couple of geometry points with an estimated influence slightly stronger than the rest of the points were located at each of the five frequencies of interest. The search of influent geometry points must be made with caution. Due to large confidence intervals of correlation coefficients with low magnitudes, see section 2.2.4, the estimated correlation coefficient can differ slightly from the actual correlation coefficient. It is therefore difficult to say with certainty that the found sensitive geometry points are the only points with influence and that the identified points in reality have that much influence as the correlation indicates. This is a problem that often occur when many variables are considered uncertain in the same analysis, see section 3.2.2. For more reliable correlation coefficients it is necessary to repeat the analysis several times and only vary a few input parameters each time. When less input variables are influent on the system response the difference between the correlation coefficients of influent and non-influent variables is larger and the confidence intervals of the important coefficient are smaller. The important input variables can then be recognised with more reliability.

In each of the important frequencies with high mean value and high variance, a couple of geometry points with slightly more influence than the other are identified. Figures 6.41, 6.42, 6.43, 6.44 and 6.45 show which geometry points affects each of the frequencies of interest.

Looking at the entire frequency interval, between 25 and 400 Hz , a total of 15 geometry points were found with a correlation coefficient larger than 0.4. They are influential in one or more frequency areas. These points are visualised in figure 6.46. Interesting is that all these points except one are points with mean value in zero, i.e. they do not belong to a band but are located in the areas between. Especially p_{34} , p_{37} and p_{44} are influential in large parts of the frequency intervals. Only 5 geometry points have a correlation to one or more responses above 0.5. Some of these high correlations are found in frequency areas with low ERP and are for that reason not particularly interesting despite the high coefficients.

The relatively large number of input variables (geometry points) with equal correlation with the ERP at the frequencies of interest means that many measures must be made to improve the robustness. The reduction of the variance in one geometry point only will not lead to an improvement of the variance of the ERP.

The objectives have to be defined more clearly to be able to derive a suitable improvement procedure. The overall mean is already robust but an improvement is possible in single smaller frequency ranges. Some of the influential geometry points show both positive and negative correlation coefficients in different frequency areas. It must be thoroughly discussed which frequencies are the most important and the improvement must be focused at these frequencies with the risk that the ERP at other frequencies increases.

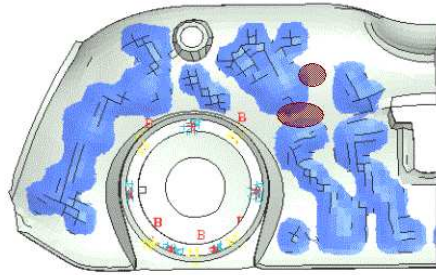


Figure 6.41: Areas especially sensitive to geometry variations (red) regarding the ERP at 58.6 Hz.

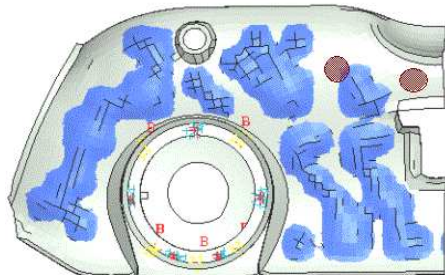


Figure 6.42: Areas especially sensitive to geometry variations (red) regarding the ERP at 74.3 Hz.

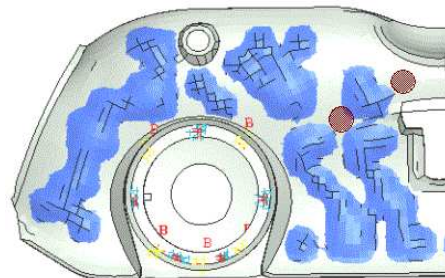


Figure 6.43: Areas especially sensitive to geometry variations (red) regarding the ERP at 125.6 Hz.

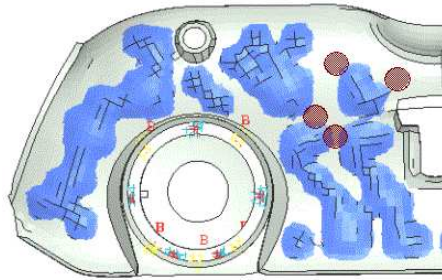


Figure 6.44: Areas especially sensitive to geometry variations (red) regarding the ERP at 162.5 Hz.

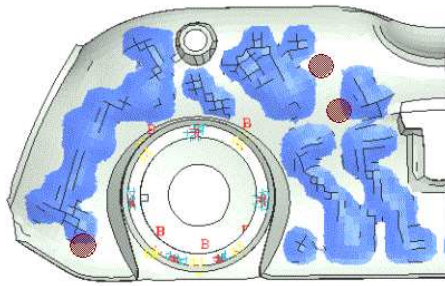


Figure 6.45: Areas especially sensitive to geometry variations (red) regarding the ERP at 369.5 Hz.

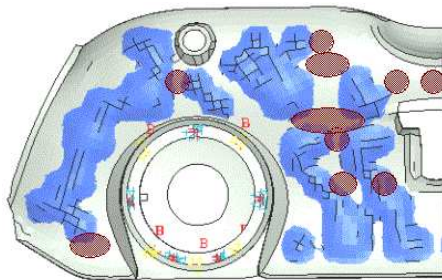


Figure 6.46: Areas especially sensitive to geometry variations (red) regarding the ERP at some part of the frequency interval.

Chapter 7

Summary

7.1 Conclusions

The robustness analysis has been divided into five steps. Different methods to perform these steps have been investigated and their suitability for the purpose of robustness analyses of FE-models have been discussed.

1. Define the objectives of the analysis

This part has to be defined specifically for each problem. For a robustness analysis, questions as '*How do my system response vary due to uncontrollable uncertainties in the parameters?*' and '*Which parameter has the largest influence on my system response?*' are common.

2. Choose the uncertain parameters

It is important to locate the variables that are uncertain in the reality. If too few variables are included, i.e. if a variable that in reality contain physical random variation is considered constant in the simulation, then the result of the robustness analysis is useless. Therefore, all variables that are expected to be uncertain in reality shall be included and varied in the analysis. It is also important to find a suitable distribution of the variables. Including variables without influence does not increase the difficulty of the analysis, many variables with strong influence can make the evaluation slightly more difficult. An increase of the number of variables increases the error of the absolute values of the statistical quantities, especially the mean value, but the error of the normalised values decreases, see the result of the tests in section 4.2.3.

3. Choose an analysis method.

For a robustness analysis of a FE-model, *stochastic sampling methods* are well suited. Test have shown that fewer samples are needed if the samples are chosen with *Latin hypercubes*. Then a sample size of 200 is recommended. The tests in section 4.2.3 show that 200 samples gives enough data for the statistical evaluation with an, in many cases, acceptable computing time.

4. Collect the data

The open source software DAKOTA [3] together with the simulation code for the specific problem suits well for robustness analyses of FE-models.

5. Evaluation

The evaluation of data collected with a sampling method are preferably evaluated with statistics. Recommended in this thesis are a *regression analysis* with a following *correlation analysis* together with a graphical analysis. The *graphical analysis* means an interpretation of different plots, for example scatter plots, histograms and response surfaces.

For the last part, the evaluation, a tool to be used with MATLAB [4] was developed. This tool calculates the necessary statistical quantities and helps the user to perform the graphical analysis. Plots can easily be created with different combinations of variables

and the most important is the ability to identify and mark single or groups of samples. The plots are linked and an individual sample can be followed in different plots.

The proposed method for the robustness analysis has been validated with two examples from the automotive industry. The stress-strain curve obtained in a compression of a foam material and the acoustic performance of a rear floor panel have been analysed for robustness. The analysis was in both cases successful and a few parameters with high influence on the variation of the system response in both cases were identified.

The two problems used for validation are different both regarding the parameters involved and the type of analysis. No reasons to suspect that the method is limited to some problems have been found. The use of a sampling method has made the method general and it can probably be implemented also on other analyses than FE-analyses, for example *Multi Body Simulation*.

7.2 Proposal for future work

For an improvement of the efficiency of the methods different actions can be investigated. Some actions are based on the stochastic sampling methods and relatively simple to implement. The effect of the accuracy of the result and restrictions on the model and simulation type must be investigated. Examples are

- *Sampling using an approximated response surface*
A few samples calculated with the real model are collected. With these samples a response surface/function is approximated and the rest of the samples for accurate statistical quantities are collected with this approximated response surface.
- *An iteration loop with integrated test of significance*
The iteration loop is automatically interrupted when enough accuracy in the statistical quantities (mainly the correlation coefficients) is achieved.

Other methods to be investigated are numerical robustness analysis methods integrated in the FE-analysis code. This can for example be *polynomial chaos* [GHANEM *et al.* 1991], the *Neumann Expansion Method* and the *Taylor Expansion Method* [OLSSON 2002].

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Software

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