PROBLEMS IN CONSTITUTIVE MODELLING

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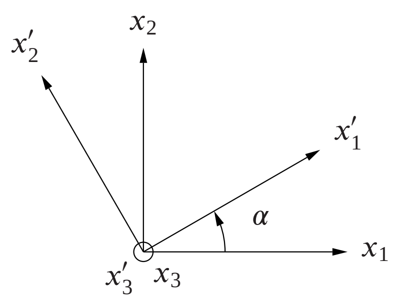
1. NOTATIONS AND CARTESIAN TENSORS

1.1. The x_i-coordinate system is transformed to the x_i'-coordinate system by the transformation matrix $\boldsymbol{A}$.

$$\boldsymbol{A} = \frac{1}{25}\begin{bmatrix} 12 & -9 & 20 \\ 15 & 20 & 0 \\ -16 & 12 & 15 \end{bmatrix}$$

Show that the point $(0, 1, -1)$ in the x_i-system coincides with the point $(-29/25, 4/5, -3/25)$ in the x_i'-system.

1.2 The original cartesian x_i-coordinate system is rotated an angle α about the x_3-axis to get a new x_i'-coordinate system, see figure.



The relation between the x_i- and the x_i'-coordinate system can be written in matrix form as

$$\boldsymbol{x} = \boldsymbol{A}^T \boldsymbol{x}'$$

a) Determine for this special case $\boldsymbol{A}^T$ and $\boldsymbol{A}$

b) Show for this special case that

$$\boldsymbol{A}^T\boldsymbol{A} = \boldsymbol{I}\ , \quad \boldsymbol{A}\boldsymbol{A}^T = \boldsymbol{I}$$

i.e., that $\boldsymbol{A}$ is an orthogonal matrix (this is a general property of $\boldsymbol{A}$).

1.3 If Φ is a scalar, show the following:

a) $\Phi_{,i}$ is a first-order tensor.

b) $\Phi_{,ij}$ is a second-order tensor.

c) $\Phi_{,kk}$ is a zero-order tensor (scalar).

1.4 Using the transformation matrix $\boldsymbol{A}$ given in **1.1**, show that the following two planes coincide:

$$2x_1 - \frac{1}{3}x_2 + x_3 = 1 \qquad \text{expressed in the } x_i\text{-system}$$

$$\frac{47}{25}x_1^{'} + \frac{14}{15}x_2^{'} - \frac{21}{25}x_3^{'} = 1 \qquad \text{expressed in the } x_i^{'}\text{-system.}$$

1.5 If $b_i = a_i/\sqrt{a_j a_j}$, show that b_i is a unit vector.

1.6 For Kronecker's delta δ_{rs} show that $\delta_{ij}\delta_{jk} = \delta_{ik}$.

1.7 Given the relations

$$\sigma_{ij} = s_{ij} + \frac{1}{3}\sigma_{kk}\delta_{ij}$$

$$J_2 = \frac{1}{2}s_{ij}s_{ji}$$

where σ_{ij} and s_{ij} are symmetric second-order tensors, show that:

a) $s_{ii} = 0$

b) $\partial J_2/\partial\sigma_{ij} = s_{ij}$

1.8 Prove that there is no pair of vectors a_i and b_i such that $a_i b_j = \delta_{ij}$.

1.9 For an arbitrary second-order tensor σ_{ij}, define s_{ij} by

$$s_{ij} = \sigma_{ij} - \alpha\delta_{ij}$$

Determine α such that $s_{ii} = 0$. The quantity s_{ij} is called the *deviatoric part* of σ_{ij}. Prove that if σ_{ij} is symmetric, also s_{ij} is symmetric.

1.10 Show that if all components of a tensor vanish in one coordinate system, then they vanish in all other coordinate systems.

1.11 Prove the theorem: The sum or difference of two tensors of the same type is again a tensor of the same type.

1.12 If a_{ij} is a tensor and the components $a_{ij} = a_{ji}$, then the tensor is called a *symmetric tensor*. If the components $a_{ij} = -a_{ji}$, then the tensor is said to be *anti-symmetric*. Show that these symmetry properties are conserved under coordinate transformations.

1.13 Hooke's law for a linear elastic material can be written

$$\sigma_{ij} = D_{ijkl}\epsilon_{kl} \tag{1}$$

where D_{ijkl} is the elastic stiffness matrix. The expression in (1) is valid in the x_i-coordinate system. If we instead express (1) in a new $x_i^{'}$-coordinate system we get

$$\sigma_{ij}^{'} = D_{ijkl}^{'}\epsilon_{kl}^{'}$$

where the x_i- and $x_i^{'}$-coordinates are related to each other by

$$x_i^{'} = A_{ij}x_j - c_i$$

and A_{ij} is the coordinate transformation matrix. This matrix fulfils the conditions

$$A_{ki}A_{kj} = \delta_{ij} \ , \qquad A_{ik}A_{jk} = \delta_{ij}$$

Since D_{ijkl} is a fourth-order tensor, the components D_{ijkl} and $D_{ijkl}^{'}$ are related by

$$D_{ijkl}^{'} = A_{im}A_{jn}D_{mnpq}A_{kp}A_{lq} \tag{2}$$

If (1) expresses Hooke's law for isotropic materials, D_{ijkl} is given as

$$D_{ijkl} = 2G\left[\frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \frac{\nu}{1-2\nu}\delta_{ij}\delta_{kl}\right] \tag{3}$$

where G is the shear modulus and ν is Poisson's ratio.
Calculate $D_{ijkl}^{'}$ using (2) and (3) and comment upon the result.

2. STRAIN TENSOR

2.1 Prove formally that the Cauchy strain invariants

$$\theta_1 = \epsilon_{ii} \qquad \theta_2 = \frac{1}{2}\theta_1^2 - \frac{1}{2}\epsilon_{ij}\epsilon_{ji} \qquad \theta_3 = \det(\epsilon_{ij})$$

are invariants.
Hint: Determine the invariants in a new coordinate system and use the transformation rules for a tensor. Moreover, since $\det \boldsymbol{A} = 1$, we have

$$\det(\boldsymbol{A}\boldsymbol{\epsilon}\boldsymbol{A}^T) = \det \boldsymbol{A} \cdot \det \boldsymbol{\epsilon} \cdot \det \boldsymbol{A} = (\det \boldsymbol{A})^2 \cdot \det \boldsymbol{\epsilon} = \det \boldsymbol{\epsilon}$$

2.2 Prove that $J_3' = \frac{1}{3}e_{ij}e_{jk}e_{ki}$ can be expressed as

$$J_3' = \frac{1}{3}(e_1^3 + e_2^3 + e_3^3)$$

2.3 The displacement field for a deformed body is given by

$$u_i = k(x_1^2,\ 2x_1x_2,\ 2x_1x_3)$$

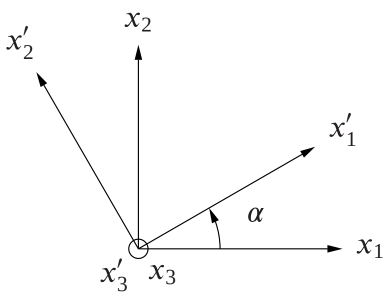
where k is a constant.

a) Determine the strain tensor ϵ_{ij}.

b) At the point (1, 2, 3) determine the tensorial shear strain and the engineering shear strain between the two directions defined by the orthogonal vectors $(3,\ 4,\ 0)$ and $(\frac{1}{3}, -\frac{1}{4}, \frac{1}{4})$. Determine also the normal strain in the direction of (3, 4, 0).

c) At the point (1, 4, 0) determine the principal strains and the corresponding principal directions.

2.4 The original cartesian x_i-coordinate system is rotated an angle α about the x_3-axis to get a new x_i'-coordinate system, see figure.



The relation between the x_i- and the x_i'-coordinate system can be written in matrix form as

$$\boldsymbol{x} = \boldsymbol{A}^T\boldsymbol{x}'$$

In this particular case, we have

$$\boldsymbol{A}^T = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

a) Since the strain tensor ϵ_{ij} is a second-order tensor, the components ϵ_{ij} in the x_i-coordinate system and the corresponding components ϵ_{ij}' in the x_i'-coordinate system are related to each other by

$$\epsilon_{kl}' = A_{ki}\epsilon_{ij}A_{lj} \qquad \text{or} \qquad \boldsymbol{\epsilon}' = \boldsymbol{A}\boldsymbol{\epsilon}\boldsymbol{A}^T$$

Suppose that ϵ_{ij} is given by

$$[\epsilon_{ij}] = \frac{10^{-4}}{2}\begin{bmatrix} 5 & 10 & 0 \\ 10 & 5 & 0 \\ 0 & 0 & -6 \end{bmatrix}$$

Calculate the components of ϵ'_{ij}.

b) From the result in a), calculate the ϵ'_{ij}-components when the angle α is chosen as α=45^o. Comment upon the result.

2.5 If the strain tensor ϵ_{ij} is given as

$$[\epsilon_{ij}] = \frac{10^{-4}}{2}\begin{bmatrix} 5 & 10 & 0 \\ 10 & 5 & 0 \\ 0 & 0 & -6 \end{bmatrix}$$

a) Determine the principal strains and the corresponding principal directions.

b) Compare the result in a) with the result in problem **2.4** b) and comment upon the result.

3. STRESS TENSOR

3.1 The deviatoric stress tensor s_{ij} is defined by

$$s_{ij} = \sigma_{ij} - \frac{1}{3}\delta_{ij}\sigma_{kk}$$

Show that the principal directions for s_{ij} and σ_{ij} coincide.

3.2 Using tensor notation, Hooke's law for general isotropic elastic behaviour can be written as

$$\epsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij} \tag{1}$$

where E = Young's modulus and ν = Poisson's ratio.

a) Do the principal directions for ϵ_{ij} and σ_{ij} coincide or not? Prove your statement.

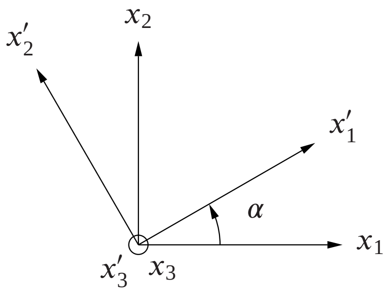
b) Using matrix notation, Eqn (1) can be written as $\boldsymbol{\epsilon} = \boldsymbol{C}\boldsymbol{\sigma}$ where

$$\boldsymbol{\epsilon}^T = (\epsilon_{11}, \epsilon_{22}, \epsilon_{33}, 2\epsilon_{12}, 2\epsilon_{13}, 2\epsilon_{23})$$

$$\boldsymbol{\sigma}^T = (\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{13}, \sigma_{23})$$

Determine the matrix $\boldsymbol{C}$.

3.3 The original cartesian x_i-coordinate system is rotated an angle α about the x_3-axis to get a new $x_i^{'}$-coordinate system, see figure.



The relation between the x_i- and the $x_i^{'}$-coordinate system can be written in matrix form as

$$\boldsymbol{x} = \boldsymbol{A}^T \boldsymbol{x}^{'}$$

In this particular case, we have

$$\boldsymbol{A}^T = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

a) Since the stress tensor σ_{ij} is a second-order tensor, the components σ_{ij} in the x_i-coordinate system and the corresponding components $\sigma_{ij}^{'}$ in the $x_i^{'}$-coordinate system are related to each other by

$$[\sigma_{kl}^{'}] = A_{ki}\sigma_{ij}A_{lj} \quad \text{or} \quad \boldsymbol{\sigma}^{'} = \boldsymbol{A}\boldsymbol{\sigma}\boldsymbol{A}^T$$

Suppose that σ_{ij} is given by

$$[\sigma_{ij}] = \begin{bmatrix} 5 & 10 & 0 \\ 10 & 5 & 0 \\ 0 & 0 & -6 \end{bmatrix}$$

Calculate the components of $\sigma_{ij}^{'}$.

b) From the result in a), calculate the $\sigma_{ij}^{'}$-components when the angle α is chosen as α=45^o. Comment upon the result.

3.4 If σ_{ij} is given as

$$[\sigma_{ij}] = \begin{bmatrix} 5 & 10 & 0 \\ 10 & 5 & 0 \\ 0 & 0 & -6 \end{bmatrix}$$

a) Determine the principal stresses and the corresponding principal directions.

b) Compare the result in a) with the result in problem **3.3** b) and comment upon the result.

3.5 Assume that the coordinate system is colinear with the principal directions of the stress tensor.

a) Determine the traction vector $\boldsymbol{t}$ on the surface, where the outer normal unit vector $\boldsymbol{n}$ is given by $\boldsymbol{n}^T = (1, 1, 1)/\sqrt{3}$.

b) The component of $\boldsymbol{t}$ along $\boldsymbol{n}$ is called the normal stress σ_n and the component of $\boldsymbol{t}$ along the surface is called the shear stress τ_n. For this particular choice of coordinate system and $\boldsymbol{n}$-vector the following notation is often employed:

$$\sigma_n = \sigma_o = \text{octahedral normal stress.}$$

$$\tau_n = \tau_o = \text{octahedral shear stress.}$$

Show that

$$\tau_n^2 = \tau_o^2 = \frac{2}{3}J_2 \quad \text{where} \quad J_2 = \frac{1}{2}tr(\boldsymbol{s}^2) = \frac{1}{2}s_{ij}s_{ji}$$

$$\sigma_n = \sigma_o = \frac{1}{3}I_1 \quad \text{where} \quad I_1 = tr\boldsymbol{\sigma} = \sigma_{kk}$$

3.6 A circular disk without any holes and made of an arbitrary material is loaded along the circular boundary by a uniform radial pressure $p_1 > 0$.

a) Prove that all boundary conditions and equilibrium conditions are fulfilled by the following stress state

$$\sigma_{11} = \sigma_{22} = -p_1 \quad \sigma_{33} = \sigma_{12} = \sigma_{13} = \sigma_{23} = 0$$

where the x_1- and x_2-axes are located in the disk plane.

b) Prove that the stress state given above is not a proper stress state, if there also exists a circular hole in the center of the disk loaded by a uniform radial pressure $p_2 > 0$, unless $p_2 = p_1$.

3.7 At a point the following stress state is given

$$\sigma_{11} = \sigma_{22} = \sigma_{12} = \sigma_{13} = \sigma_{23} = p \quad \text{and} \quad \sigma_{33} = 0$$

a) By hand-calculations, determine the principal stresses and directions and check that the principal directions are orthogonal.

b) Same as a), but now use CALFEM or another program; compare the results.

c) The original coordinate system is given by the x_i-coordinates. A new x_i'-coordinate system is chosen to be colinear with the principal directions. The relation between the new and old coordinates can be written as $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$. Determine $\boldsymbol{A}$ and show that for the particular $\boldsymbol{A}$-matrix in question, we have $\boldsymbol{A}^T\boldsymbol{A} = \boldsymbol{I}$, i.e. $\boldsymbol{A}$ is an orthogonal matrix (this is a general property of $\boldsymbol{A}$).

d) Demonstrate by inspection that the $\boldsymbol{\sigma}$-matrix given above fulfils Cayley-Hamilton's theorem.

3.8 In the x_i-coordinate system an arbitrary stress matrix $\boldsymbol{\sigma}$ is given. A new x_i'-coordinate system is chosen, so that the unit vectors along the x_1'-, x_2'- and x_3'-axes are given by the components $\begin{bmatrix} 1 & 1 & 0 \end{bmatrix}/\sqrt{2}$, $\begin{bmatrix} -1 & 1 & 0 \end{bmatrix}/\sqrt{2}$ and $\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$, respectively, in the old coordinate system.
Determine the stress matrix $\boldsymbol{\sigma}'$ in the new coordinate system, which corresponds to $\boldsymbol{\sigma}$ in the old coordinate system.

3.9 For the unit vector $\mathbf{n}$ and the parameter k, we can define a stress tensor $\boldsymbol{\sigma}$ given by

$$\boldsymbol{\sigma} = k\boldsymbol{n}\boldsymbol{n}^T$$

Show that this stress tensor corresponds to pure tension in the direction $\mathbf{n}$.

3.10 A stress tensor is given by

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} & 2 & 1 \\ 2 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}$$

Determine σ_{11} so that a section will exist on which the traction vector $\mathbf{t}$ is zero. Determine a unit vector normal to that section.

4. HYPER–ELASTICITY

4.1 Prove the following relations

$$\frac{\partial \tilde{I}_1}{\partial \epsilon_{ij}} = \delta_{ij} \quad , \qquad \frac{\partial \tilde{I}_2}{\partial \epsilon_{ij}} = \epsilon_{ij} \quad , \qquad \frac{\partial \tilde{I}_3}{\partial \epsilon_{ij}} = \epsilon_{ik}\epsilon_{kj}$$

4.2 Determine the results of the partial derivatives

$$\frac{\partial I_1}{\partial \sigma_{ij}} \quad , \qquad \frac{\partial J_2}{\partial \sigma_{ij}} \quad , \qquad \frac{\partial J_3}{\partial \sigma_{ij}}$$

4.3 Let D_{ijkl} be given by

$$D_{ijkl} = 2G\left[\frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \frac{\nu}{1-2\nu}\delta_{ij}\delta_{kl}\right]$$

Show that $\sigma_{ij} = D_{ijkl}\epsilon_{kl}$ then implies

$$\sigma_{ij} = 2G\left[\epsilon_{ij} + \frac{\nu}{1-2\nu}\delta_{ij}\epsilon_{kk}\right]$$

Demonstrate also that D_{ijkl} fulfils the symmetry properties $D_{ijkl} = D_{jikl}$, $D_{ijkl} = D_{ijlk}$ and $D_{ijkl} = D_{klij}$.

4.4 For linear isotropic elasticity we have

$$D_{ijkl} = 2G\left[\frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \frac{\nu}{1-2\nu}\delta_{ij}\delta_{kl}\right]$$

and

$$C_{ijkl} = \frac{1}{2G}\left[\frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) - \frac{\nu}{1+\nu}\delta_{ij}\delta_{kl}\right]$$

Show that these expressions fulfil the equation

$$D_{ijmn}C_{mnkl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

4.5 We have $\epsilon_{ij} = C_{ijkl}\,\sigma_{kl}$. Multiply with D_{pqij} and show that we obtain $\sigma_{ij} = D_{ijkl}\,\epsilon_{kl}$.

4.6 We have $\sigma_{ij} = D_{ijkl}\,\epsilon_{kl}$. Write this equation in matrix form $\boldsymbol{\sigma} = \boldsymbol{D}\boldsymbol{\epsilon}$ and identify all terms. For isotropic elasticity show that $\boldsymbol{D}$ is given by

$$\boldsymbol{D} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}(1-2\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}(1-2\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(1-2\nu) \end{bmatrix}$$

4.7 Use superposition of simple loading situations to show that the strain-stress relation for an isotropic material can be written as

$$\boldsymbol{\epsilon} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu) \end{bmatrix}\boldsymbol{\sigma}$$

Hint, to obtain the relations between shear strains and stresses use can be made of Mohr's circles of stress and strain.

4.8 For a general isotropic hyper-elastic material the strain energy can be written in term of the strain invariants, i.e.

$$W = W(\tilde{I}_1, \tilde{I}_2, \tilde{I}_3)$$

where

$$\tilde{I}_1 = \epsilon_{kk}, \quad \tilde{I}_2 = \frac{1}{2}\epsilon_{ij}\epsilon_{ji}, \quad \tilde{I}_3 = \frac{1}{3}\epsilon_{ik}\epsilon_{kj}\epsilon_{ji}$$

a) Determine the strain-stress relation based on

$$\sigma_{ij} = \frac{\partial W}{\partial \epsilon_{ij}}$$

b) Based on a) and that

$$\frac{\partial W}{\partial \tilde{I}_1} = \lambda \epsilon_{kk}$$

determine that fourth-order tensor D^s_{ijkl} given by

$$\sigma_{ij} = D^s_{ijkl}\epsilon_{kl}$$

c) Derive the incremental relation

$$d\sigma_{ij} = D^t_{ijkl} d\epsilon_{kl} \qquad \text{where} \qquad D^t_{ijkl} = \frac{\partial^2 W}{\partial \epsilon_{ij} \partial \epsilon_{kl}}$$

and identify D^t_{ijkl}. Assume that

$$\frac{\partial W}{\partial \tilde{I}_2} \quad \text{and} \quad \frac{\partial W}{\partial \tilde{I}_3} \quad \text{are constants}$$

d) In a uniaxial loading situation, i.e. in a $\sigma - \epsilon$-graph, identify D^s and D^t (the corresponding uniaxial quantities). No calculations are necessary, a graphical illustration is sufficient.

4.9 The strain energy can be written as $W = \frac{1}{2}\sigma_{ij}\epsilon_{ij}$. Show that it also can be written as $W = \frac{1}{2}\boldsymbol{\sigma}^T\boldsymbol{\epsilon} = \frac{1}{2}\boldsymbol{\epsilon}^T\boldsymbol{\sigma}$.

4.10 A hyper-elastic material is assumed to possess the following complementary energy

$$C = aJ_2 + bI_1J_2 \tag{1}$$

where a and b are constants.

a) Derive the constitutive relation $\epsilon_{ij} = \epsilon_{ij}(\sigma_{kl})$.

b) For uniaxial tension, the stress-strain behaviour as experienced in the laboratory can be approximated by

$$\epsilon = 10^{-4}\sigma + 10^{-5}\sigma^2$$

where σ is measured in MPa. Determine the parameters a and b in (1).

c) For loading in pure shear where $\tau = 100$ MPa calculate the corresponding strains.

d) For the load defined in c), determine the volumetric strain ϵ_{kk} and compare with the volumetric strain for a linear elastic isotropic material.

4.11 Using two perpendicular symmetry planes show that the general (orthotropic) hyper-elastic stress-strain relation can be written as

$$\boldsymbol{\sigma} = \begin{bmatrix} D_{11} & D_{12} & D_{13} & 0 & 0 & 0 \\ D_{21} & D_{22} & D_{23} & 0 & 0 & 0 \\ D_{31} & D_{32} & D_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & D_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & D_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & D_{66} \end{bmatrix} \boldsymbol{\epsilon}$$

What will the use of a third symmetry plane yield?

4.12 Similar to (4.7), use simple loading and engineering definitions for the elastic moduli that the othotropic strain-stress relation can be written as

$$\boldsymbol{\epsilon} = \begin{bmatrix} \frac{1}{E_x} & -\frac{\nu_{yx}}{E_y} & -\frac{\nu_{zx}}{E_z} & 0 & 0 & 0 \\ -\frac{\nu_{xy}}{E_x} & \frac{1}{E_y} & -\frac{\nu_{zy}}{E_z} & 0 & 0 & 0 \\ -\frac{\nu_{xz}}{E_x} & -\frac{\nu_{yz}}{E_y} & \frac{1}{E_z} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{xy}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{xz}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{yz}} \end{bmatrix} \boldsymbol{\sigma}$$

Give also a physical interpretation of the elastic constants, as well as, assuming hyper-elasticity comment upon the symmertry reqirements.

4.13 Derive the constitutive matrix $\boldsymbol{D}$ corresponding to plane stress for an orthotropic material.

5. CAUCHY - ELASTICITY

5.1 Consider isotropic hyper-elasticity. The most general format of the complementary energy C is then given by

$$C = C(I_1,\ J_2,\ J_3) \tag{1}$$

a) Derive the most general hyper-elasticity format $\epsilon_{ij} = \epsilon_{ij}(\sigma_{kl})$.

b) Write the most general non-linear Hooke formulation for hyper-elasticity expressed in terms of the non-linear parameters G and K. Can G and K depend on J_3?

c) Consider isotropic Cauchy-elasticity. What is the most general Cauchy-elasticity format $\epsilon_{ij} = \epsilon_{ij}(\sigma_{kl})$?

d) Write the most general non-linear Hooke formulation for Cauchy-elasticity expressed in terms of the nonlinear parameters G and K. Can G and K depend on J_3?

e) For which materials are the influence of J_3 important?

f) For nonlinear elasticity, what is the difference in response during loading and unloading? Which materials behave like nonlinear elasticity during unloading?

6. REPRESENTATION THEOREMS

6.1 Prove by inspection that

$$\boldsymbol{\epsilon} = \alpha_1 \boldsymbol{I} + \alpha_2 \boldsymbol{\sigma} + \alpha_3 \boldsymbol{\sigma}^2$$

satisfies the conditions of coordinate invariance and isotropy.

6.2 Write in index notation the tensor generators $\mathbf{G}_1 \ldots \mathbf{G}_8$ given by

$$\boldsymbol{G}_1 = \boldsymbol{I}\ ;\ \ \boldsymbol{G}_2 = \boldsymbol{N}\ ;\ \ \boldsymbol{G}_3 = \boldsymbol{N}^2\ ;\ \ \boldsymbol{G}_4 = \boldsymbol{P}\ ;\ \ \boldsymbol{G}_5 = \boldsymbol{P}^2$$
$$\boldsymbol{G}_6 = \boldsymbol{N}\boldsymbol{P} + \boldsymbol{P}\boldsymbol{N}\ ;\ \ \boldsymbol{G}_7 = \boldsymbol{N}^2\boldsymbol{P} + \boldsymbol{P}\boldsymbol{N}^2\ ;$$
$$\boldsymbol{G}_8 = \boldsymbol{N}\boldsymbol{P}^2 + \boldsymbol{P}^2\boldsymbol{N}$$

6.3 Derive the constitutive relation for the Maxwell model and the Kevin model and determine the responses for a sudden applied load for the uniaxial loading situation.

7. HYPO - ELASTICITY

7.1 Equation

$$\begin{aligned}\dot{\sigma}_{ij} \;=\;& \beta_1\dot{\epsilon}_{kk}\delta_{ij} + \beta_2\dot{\epsilon}_{ij} + \beta_3\dot{\epsilon}_{kk}\sigma_{ij} + \beta_4\sigma_{mn}\dot{\epsilon}_{mn}\delta_{ij} \\ & +\beta_5(\sigma_{ik}\dot{\epsilon}_{kj} + \dot{\epsilon}_{ik}\sigma_{kj}) + \beta_6\dot{\epsilon}_{mm}\sigma_{ik}\sigma_{kj} + \beta_7\sigma_{mn}\dot{\epsilon}_{nm}\sigma_{ij} \\ & +\beta_8\sigma_{lm}\sigma_{mn}\dot{\epsilon}_{nl}\delta_{ij} + \beta_9(\sigma_{ik}\sigma_{kl}\dot{\epsilon}_{lj} + \dot{\epsilon}_{ik}\sigma_{kl}\sigma_{lj}) \\ & +\beta_{10}\sigma_{mn}\dot{\epsilon}_{nm}\sigma_{ik}\sigma_{kj} + \beta_{11}\sigma_{lm}\sigma_{mn}\dot{\epsilon}_{nl}\sigma_{ij} \\ & +\beta_{12}\sigma_{lm}\sigma_{mn}\dot{\epsilon}_{nl}\sigma_{ik}\sigma_{kj}\end{aligned}$$

may be written as $\dot{\sigma}_{ij} = D_{ijst}\dot{\epsilon}_{st}$. Determine the tensor D_{ijst}.

8. FAILURE AND INITIAL YIELD CRITERIA

8.1 For the stress tensor σ_{ij} given by:

$$\begin{bmatrix} \frac{3}{2} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} & \frac{11}{4} & -\frac{5}{4} \\ -\frac{1}{2\sqrt{2}} & -\frac{5}{4} & \frac{11}{4} \end{bmatrix} \text{MPa}$$

a) Find the principal stresses and the corresponding principal directions.

b) Find the deviatoric stress tensor, s_{ij}, and the principal deviatoric stresses s_1, s_2 and s_3.

c) Determine the deviatoric stress invariants J_1, J_2 and J_3.

8.2 Answer the following questions and explain the answer.

a) If $s_1 > s_2 > s_3$, can s_3 be equal to zero?

b) Can J_2 be negative?

c) Can J_3 be positive?

8.3 Show that subtracting a hydrostatic stress from a given state of stress does not change the principal directions.

8.4 The stress state at a point is given by

$$[\sigma_{ij}] = \begin{bmatrix} 30 & 45 & 60 \\ 45 & 20 & 50 \\ 60 & 50 & 10 \end{bmatrix} MPa$$

Determine the stress invariants I_1, J_2, J_3 and the Lode angle θ.

8.5 A metal yields when the maximum shear stress, τ_{max}, reaches the value of 125 MPa. A material element of this metal is subjected to a biaxial state of stress:

$$\sigma_1 = \sigma\,; \quad \sigma_2 = \alpha\sigma\,; \quad \sigma_3 = 0$$

where α is a constant and σ is positive. For what values of (σ, α) will yielding occur?

8.6 For biaxial stress states and adopting the Tresca criterion, draw the yield curve in the $\sigma_1\sigma_2$-coordinate system. Note that in this coordinate system, the usual convention of $\sigma_1 \geq \sigma_2 \geq \sigma_3$ is abandoned.

8.7 A metal yields at a state of plane stress with

$$\sigma_{11} = 80 \text{ MPa}, \quad \sigma_{22} = 40 \text{ MPa}, \quad \sigma_{12} = 80 \text{ MPa}$$

Assume isotropy, independence of hydrostatic pressure, and equality of properties for reversed loading (for instance, that tension and compression gives the same yield stress).

a) Derive other biaxial states of stress at yield in the (σ_1, σ_2)-space using the above informations.

b) Plot the result in part a) in the (σ_1, σ_2)-space and estimate the yield stress in axial tension and in simple shear, respectively, and give limits of possible error of your estimate, based on convexity.

c) Determine the yield stresses in b), based on the von Mises criterion and the Tresca criterion, respectively.

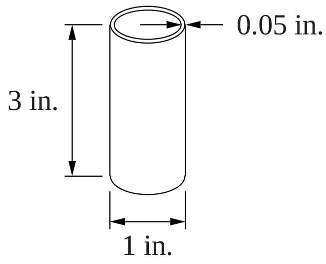
8.8 A long circular steel tube having a mean diameter of 254 mm and 3.2 mm wall thickness is subjected to an internal pressure of 4.83 MPa. The ends of the tube are closed. The yield stress of the steel is 227 MPa. Find the additional axial tensile load F which is needed to cause yielding of the tube, based on the von Mises criterion and the Tresca criterion, respectively.

8.9 a) The Coulomb failure criterion can be written as (assume $k > 1$)

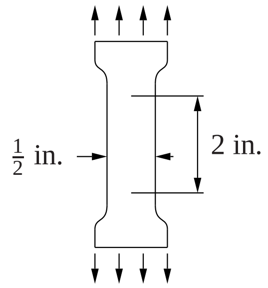
$$k\sigma_1 - \sigma_3 = \sigma_c \tag{1}$$

where $\sigma_1 \geq \sigma_2 \geq \sigma_3$ are the principal stresses (tension is considered as positive)
For plane stress conditions and considering only principal stresses show the failure curve corresponding to (1) in the $\sigma_1\sigma_2$-coordinate system (where the convention $\sigma_1 \geq \sigma_2 \geq \sigma_3$ now is abandoned). Express the uniaxial tensile strength in terms of σ_c and k.

8.10 A simple torsion test of certain material, using a hollow cylinder specimen as shown in the figure, shows that the load-deflection curve is linear for a shearing stress below 125 MPa and that at the stress 125 MPa yielding occurs. If the von Mises yield criterion is adopted, what is the expected value of the tensile stress at which yielding occur in a tension test specimen as shown in the figure?



Shear test



Tension test

8.11 A closed-ended thin-walled tube of thickness t and mean radius r is subjected to an axial tensile force F, which is less than the value F_o necessary to cause yielding. If a gradually increasing internal pressure p is now applied, show that the tube will yield according to the Tresca criterion when

$$\frac{pr}{t\sigma_{y0}} = \begin{cases} 1 & \text{when } \dfrac{F}{F_o} \leq \dfrac{1}{2} \\ 2(1 - \frac{F}{F_o}) & \text{when } \dfrac{F}{F_o} \geq \dfrac{1}{2} \end{cases}$$

and according to the von Mises criterion when

$$\frac{pr}{t\sigma_{y0}} = \frac{2}{\sqrt{3}}\sqrt{1 - (\frac{F}{F_o})^2}$$

8.12 Given the yield stresses σ_t and σ_c in uniaxial tension and compression, respectively, find the yield stress in shear resulting from the following yield criteria: a) Coulomb, b) Drucker-Prager, c) von Mises and d) Tresca.

8.13 Show that for a state of plane stress with $\sigma_{11} = \sigma$, $\sigma_{12} = \tau$ and $\sigma_{22} = 0$, both the Tresca and von Mises yield criterion can be expressed in the form

$$(\frac{\sigma}{\sigma_{yo}})^2 + (\frac{\tau}{\tau_{y0}})^2 = 1$$

How is σ_{y0} and τ_{y0} related for the Tresca and the von Mises criterion, respectively?

8.14 The initial yield criterion of Drucker-Prager is defined by

$$f = \sqrt{3J_2} + \alpha I_1 - \beta = 0$$

where α and β are parameters and

$$J_2 = \frac{1}{2}s_{ij}s_{ij} \qquad s_{ij} = \sigma_{ij} - \frac{1}{3}\delta_{ij}\sigma_{kk} \qquad I_1 = \sigma_{kk}$$

Consider the stress state

$$[\sigma_{ij}] = \begin{bmatrix} \sigma & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

a) Assume that loading takes place such that $\tau = \sigma$. Calculate the value of σ for which yielding starts. Both $\sigma > 0$ and $\sigma < 0$ should be considered.

b) In the meridian plane, $\sqrt{3J_2} - I_1$, draw the shape of the Drucker-Prager yield criterion and the loading paths given by $\tau = \sigma$. Both $\sigma > 0$ and $\sigma < 0$ should be considered.

c) In the deviatoric plane illustrate the shape of the Drucker-Prager yield criterion and the loading path given by $\sigma = 0$ and $\tau \neq 0$. (one path is sufficient). Hint: the angle is given by

$$\cos(3\theta) = \frac{3\sqrt{3}}{2}\frac{J_3}{J_2^{3/2}} \quad \text{where} \quad J_3 = \frac{1}{3}s_{ik}s_{kj}s_{ji}$$

8.15 von Mises isotropic criterion can be written as

$$s_{ij}P_{ijkl}s_{kl} - 1 = 0 \qquad \text{or} \quad \boldsymbol{s}^T\boldsymbol{P}\boldsymbol{s} - 1 = 0$$

where

$$P_{ijkl} = \frac{3}{4\sigma_{yo}^2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

Derive the matrix $\boldsymbol{P}$ corresponding to P_{ijkl}. Next, use that $s_{ii} = 0$ to derive an alternative format of the von Mises isotropic criterion, i.e.

$$\boldsymbol{s}^T\hat{\boldsymbol{P}}\boldsymbol{s} - 1 = 0 \qquad \text{where} \quad \hat{\boldsymbol{P}} \neq \boldsymbol{P}$$

8.16 Show that

$$\boldsymbol{\sigma}^T\boldsymbol{P}\boldsymbol{\sigma} - 1 = 0$$

for the special choice

$$\boldsymbol{P} = \begin{bmatrix} F+G & -F & -G & 0 & 0 & 0 \\ -F & F+H & -H & 0 & 0 & 0 \\ -G & -H & G+H & 0 & 0 & 0 \\ 0 & 0 & 0 & 2L & 0 & 0 \\ 0 & 0 & 0 & 0 & 2M & 0 \\ 0 & 0 & 0 & 0 & 0 & 2N \end{bmatrix}$$

can be written as

$$\boldsymbol{s}^T \boldsymbol{P} \boldsymbol{s} - 1 = 0$$

8.17 Derive the general format of $\boldsymbol{P}$ and $\boldsymbol{q}$ in

$$\boldsymbol{\sigma}^T \boldsymbol{P} \boldsymbol{\sigma} - \boldsymbol{q}^T \boldsymbol{\sigma} - 1 = 0$$

if orthotropy and pressure independent response is assumed. How may independent parameters does the model have?

8.18 The von Mises criterion for orthotropy can be written as $\boldsymbol{\sigma}^T \boldsymbol{P} \boldsymbol{\sigma} - 1 = 0$ where

$$\boldsymbol{P} = \begin{bmatrix} A & -F & -G & 0 & 0 & 0 \\ -F & B & -H & 0 & 0 & 0 \\ -G & -H & C & 0 & 0 & 0 \\ 0 & 0 & 0 & 2L & 0 & 0 \\ 0 & 0 & 0 & 0 & 2M & 0 \\ 0 & 0 & 0 & 0 & 0 & 2N \end{bmatrix}$$

Establish (imagined) test methods such that all parameters can be determined.

9. INTRODUCTION TO PLASTICITY THEORY

9.1 Let the effective plastic strain rate $\dot{\epsilon}^p_{eff}$ and the effective stress σ_{eff} be defined by

$$\dot{\epsilon}^p_{eff} = (\frac{2}{3}\, \dot{\epsilon}^p_{ij}\, \dot{\epsilon}^p_{ij})^{1/2}\ ; \qquad \sigma_{eff} = (\frac{3}{2}\, s_{ij}\, s_{ij})^{1/2}$$

Show that we then have the convenient properties that $\sigma_{eff} = \sigma$ holds for uniaxial tension and that $\dot{\epsilon}^p_{eff} = \dot{\epsilon}^p$ holds for uniaxial tension if plastic incompressibility is assumed.

9.2 A von Mises material is considered. For isotropic hardening, we have

$$f = (\frac{3}{2} s_{ij} s_{ij})^{1/2} - \sigma_y = 0$$

whereas kinematic hardening is given by

$$f = [\frac{3}{2}(s_{ij} - \alpha^d_{ij})(s_{ij} - \alpha^d_{ij})]^{1/2} - \sigma_{y0} = 0$$

For associated plasticity, derive expressions for $\dot{\epsilon}^p_{ij}$.

9.3 Ideal plasticity according to the following criteria is considered:

a) the von Mises yield criterion: $\sqrt{3J_2} - \sigma_{yo} = 0$

b) the Tresca yield criterion: $\sigma_1 - \sigma_3 - \sigma_{yo} = 0$; $\quad \sigma_1 \geq \sigma_2 \geq \sigma_3$

c) the Drucker - Prager criterion: $\sqrt{3J_2} + \alpha I_1 - \beta = 0$

d) the Coulomb criterion: $k\sigma_1 - \sigma_3 - \sigma_{y0} = 0$; $\quad \sigma_1 \geq \sigma_2 \geq \sigma_3$

A material element is subjected to proportional loading. The principal stresses are given by (2σ, σ, 0) where σ is an increasing stress value. Find the magnitude of σ where the material begins to yield. Adopting the associated flow rule, find also the plastic strain rate $\dot{\epsilon}^p_{ij}$ at onset of yielding expressed in terms of the plastic multiplier $\dot{\lambda}$. If the effective plastic strain rate $\dot{\epsilon}^p_{eff}$ is defined as $\dot{\epsilon}^p_{eff} = (\frac{2}{3}\dot{\epsilon}^p_{ij}\dot{\epsilon}^p_{ij})^{1/2}$, how is $\dot{\lambda}$ related to $\dot{\epsilon}^p_{eff}$?

9.4 The same problem as 9.3, but for the principal stresses (σ, σ, 0) and only considering the von Mises and the Drucker-Prager criteria. When considering the Tresca and the Coulomb criteria, what problem is encountered if you should determine $\dot{\epsilon}^p_{ij}$?

9.5 Isotropic hardening of a von Mises material is given by

$$f(\sigma_{ij}, K) = \sqrt{3J_2} - \sigma_{yo} - K(\kappa) = 0 \qquad (1)$$

where

$$J_2 = \frac{1}{2}s_{kl}s_{kl} \quad ; \qquad s_{kl} = \sigma_{kl} - \frac{1}{3}\delta_{kl}\sigma_{pp}$$

and σ_{yo} is the initial yield stress in tension. The current yield stress σ_y is then given by

$$\sigma_y(\kappa) = \sigma_{yo} + K(\kappa)$$

i.e. (1) takes the form

$$f = \sqrt{3J_2} - \sigma_y(\kappa) = 0 \qquad (2)$$

The associated flow rule provides

$$\dot{\epsilon}^p_{ij} = \dot{\lambda}\frac{\partial f}{\partial \sigma_{ij}} \quad ; \qquad \dot{\lambda} \geq 0 \qquad (3)$$

The effective plastic strain rate is defined by

$$\dot{\epsilon}^p_{eff} = (\frac{2}{3}\dot{\epsilon}^p_{ij}\dot{\epsilon}^p_{ij})^{1/2} \quad (4)$$

Strain hardening is assumed, i.e. the following evolution law for κ is assumed

$$\dot{\kappa} = \dot{\epsilon}^p_{eff}$$

a) Based on (2) and (3) determine the explicit form of the plastic strain rate, i.e. $\dot{\epsilon}^p_{ij}$.

b) From the definition (4) prove that we have

$$\dot{\epsilon}^p_{eff} = \dot{\lambda} \quad (5)$$

c) Consider uniaxial tension where

$$[\sigma_{ij}] = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (6)$$

For this stress state prove that yielding requires that

$$\sigma = \sigma_y(\epsilon^p_{eff})$$

d) For the stress state defined by (6), identify the expression

$$\dot{\epsilon}^p_{ij} = \dot{\lambda} \begin{bmatrix} ? \\ 3 \times 3 \end{bmatrix}$$

Denote $\dot{\epsilon}^p$ by $\dot{\epsilon}^p = \dot{\epsilon}^p_{11}$ and show that in the present case, we have

$$\dot{\epsilon}^p_{eff} = \dot{\epsilon}^p$$

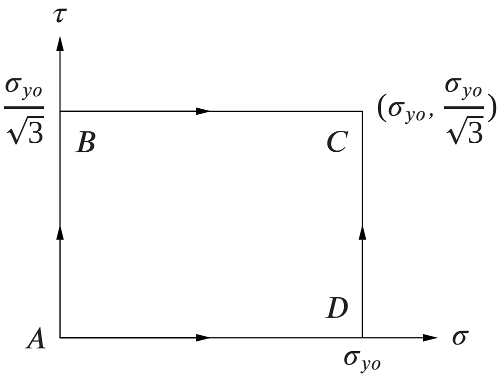
9.6 The initial yield stress is σ_{y0} and during uniaxial tension in the plastic regime, we have $d\sigma/d\epsilon_p = H$ where H is a constant (i.e. linear hardening). Investigate isotropic hardening of a von Mises material. Associated plasticity is adopted. Calculate the resulting elastic and plastic strains at point C for the load histories:

a) load path ABC

b) load path ADC

c) proportial loading, i.e. load path AC

d) calculate the curve $\sigma_{eff} = \sigma_{eff}(\epsilon^p_{eff})$ for the three load cases mentioned above and comment upon the result.

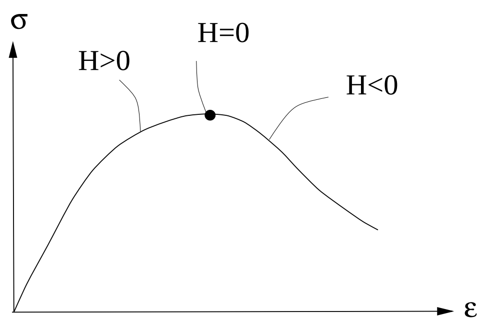


Note: $\int \frac{x^2\,dx}{a+bx^2} = \frac{x}{b} - \frac{a}{b}\frac{1}{\sqrt{ab}}\arctan\frac{x\sqrt{ab}}{a}$

10. GENERAL PLASTICITY THEORY

10.1 Write all nesessary equations to establish $\dot{\sigma}_{ij} = D^{ep}_{ijkl}\dot{\epsilon}_{kl}$.

10.2 The interpretation of the plastic modulus in the uniaxial case is shown in the figure below.



Show that this is true for all plasticity models.

12. COMMON PLASTICITY MODELS

12.1 From $\dot{\sigma}_{ij} = D^{ep}_{ijkl}\dot{\epsilon}_{kl}$ where

$$D^{ep}_{ijkl} = D_{ijkl} - \frac{9G^2}{A} s_{ij}s_{kl}$$

and

$$D_{ijkl} = 2G[\frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \frac{\nu}{1-2\nu}\delta_{ij}\delta_{kl}]$$

derive the corresponding matrix format, i.e. $\dot{\boldsymbol{\sigma}} = \boldsymbol{D}^{ep}\dot{\boldsymbol{\epsilon}}$.

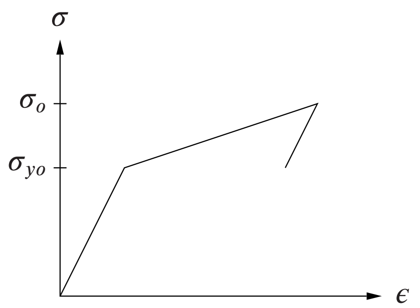
12.2 a) Derive the plane strain formula of $\dot{\boldsymbol{\sigma}} = \boldsymbol{D}^{ep}\dot{\boldsymbol{\epsilon}}$ in (12.1). **b)** Determine also the expression for the out-of-plane stress rates $\dot{\sigma}_{13}$, $\dot{\sigma}_{23}$ and $\dot{\sigma}_{33}$.

12.3 Using the von Mises model with the initial yield stress $\sigma_{y\circ}$, the following loading test is conducted:

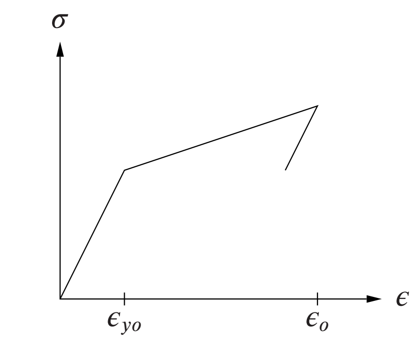
$$(\sigma, \tau) = (0, 0) \rightarrow (2\sigma_{y\circ}, 0) \rightarrow (0, 2\sigma_{y\circ}) \rightarrow (2\sigma_{y\circ}, 2\sigma_{y\circ})$$

Assuming that the behaviour of this material follows the isotropic hardening rule, draw the initial yield surface and the subsequent yield surfaces in the $\sigma - \tau$ space at the ends of the loading paths. Note that in each loading step, the load is varied proportionally.

12.4 Show how i) the isotropic and ii) the kinematic von Mises bilinear model behaves in uniaxial loading when
a) the stress is cycled 4 times between σ_o and -σ_o.



b) the strain is cycled 4 times between ϵ_o and -ϵ_o



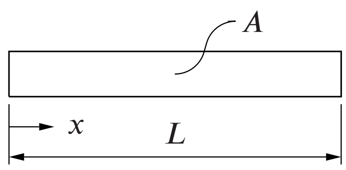
12.5 Using the Drucker-Prager criterion as the yield function $f(\sigma_{ij}, K)$, and the von Mises criterion as the plastic potential function $g(\sigma, K)$ in

$$\dot{\epsilon}_{ij}^{p} = \dot{\lambda}\frac{\partial g}{\partial \sigma_{ij}}$$

derive the expression for the scalar $\dot{\lambda}$.

15. SOLUTION OF GLOBAL EQUATIONS

15.1 Euler forward scheme and Newton-Raphson equilibrium iterations.
Consider a bar with constant area A and length L.
No body forces act on the bar.



The equilibrium condition states that

$$\frac{d}{dx}(A\sigma) = 0 \tag{1}$$

a) Show that the weak form of (1) is given by

$$\int_0^L \frac{dv}{dx} A\sigma dx = [vA\sigma]_0^L \tag{2}$$

where v is an arbitrary weight function.

b) The axial displacement $u = u(x)$ (measured positive in the x-direction) is approximated by

$$u = \boldsymbol{Na}$$

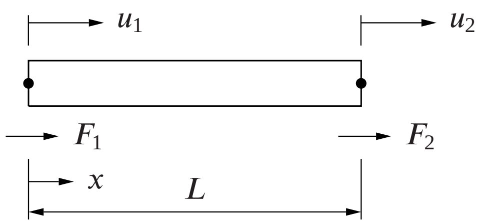
i.e. the axial strain $\epsilon = \epsilon(x)$ becomes

$$\epsilon = \boldsymbol{Ba} \qquad \text{where} \qquad \boldsymbol{B} = \frac{d\boldsymbol{N}}{dx}$$

Use the Galerkin method to express the equilibrium condition (2) as

$$\boldsymbol{\psi} = [\boldsymbol{N}^T A\sigma]_0^L - A\int_0^L \boldsymbol{B}^T \sigma dx\,; \qquad \boldsymbol{\psi} = \boldsymbol{0} \tag{3}$$

c) As indicated, the behaviour of the bar is approximated by one linear finite element. The figure shows the nodal displacements u_1 and u_2 as well as the external forces F_1 and F_2 acting on the nodal points.



Show that the equilibrium condition (3) then takes the form

$$\boldsymbol{\psi} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} - \frac{A}{L} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \int_0^L \sigma dx \, ; \qquad \boldsymbol{\psi} = \mathbf{0} \tag{4}$$

d) The behaviour of the material is assumed to be given by the constitutive equation

$$\dot{\sigma} = E^{ep} \dot{\epsilon} \tag{5}$$

where

$$E^{ep} = E(1 - 2\alpha\epsilon) \tag{6}$$

and E is Young's modulus whereas α is a dimensionless positive parameter. With this material model show that the incremental form of (4) takes the form

$$\frac{AE^{ep}}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{bmatrix} = \begin{bmatrix} \dot{F}_1 \\ \dot{F}_2 \end{bmatrix} \tag{7}$$

(Hint: how does ϵ vary along the bar?; how does E^{ep} vary along the bar?)

e)

The boundary conditions are given by

$$u_1 = 0 \, ; \qquad F_2 \text{ prescribed} \tag{8}$$

Show that the part of the incremental formulation (7) that is of interest becomes

$$\frac{AE^{ep}}{L} \dot{u}_2 = \dot{F}_2 \tag{9}$$

Discuss how and (7) and (9) are related to (13.23).
Likewise, show that the part of the equilibrium condition (4) that is of interest becomes

$$\psi = F_2 - A\sigma \,; \quad \psi = 0 \tag{10}$$

Discuss how (4) and (10) are related to (13.15) and (13.16).

f) With the definitions

$$K = \frac{AE^{ep}}{L} \,; \qquad a = u_2 \,; \qquad f = F_2 \tag{11}$$

the incremental formulation (9) can be written

$$K\dot{a} = \dot{f} \tag{12}$$

whereas the equilibrium condition (10) takes the form

$$\psi = f - A\sigma \,; \quad \psi = 0 \tag{13}$$

We assume $\alpha = 10^2$.

g) Using the Euler forward scheme, determine the result for two load steps $f_1 = 10^{-3}AE$ and $f_2 = 2 \cdot 10^{-3}AE$.
(Result: $a_1 = 10^{-3}L$; $a_2 = 2.25 \cdot 10^{-3}L$).

h) Determine the correct response by integrating (5) exactly.
(Result: $a_1 = 1.127 \cdot 10^{-3}L$; $a_2 = 2.764 \cdot 10^{-3}L$).

i) Using the Newton-Raphson approach with 3 equilibrium iterations in each load step obtain the response (4 digits-calculations).
(Result: $a_1^1 = 10^{-3}L$, $a_1^2 = 1.125 \cdot 10^{-3}L$, $a_1^3 = \underline{1.127 \cdot 10^{-3}L}$
$a_2^1 = 2.418 \cdot 10^{-3}L$, $a_2^2 = 2.741 \cdot 10^{-3}L$, $a_2^3 = \underline{2.765 \cdot 10^{-3}L}$).

j) Show the results of g), h) and i) in a figure and comment upon the results.

16 INTEGRATION OF ELASTO-PLASTIC CONSTITUTIVE EQUATIONS

16.1 For the plane stress case the von Mises yield surface can be expressed in terms of principal stresses as

$$f = \sigma_e - \sigma_{y0} = \left[\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2\right]^{1/2} - \sigma_{y0}$$

The contact state γ_c, i.e. where the stress path intersect with the yield surface can be determined from the condition

$$f = \left[(\sigma_1^c)^2 + (\sigma_2^c) + \sigma_1^c\sigma_2^c\right]^{1/2} - \sigma_{y0} = 0$$

where

$$\sigma_{ij}^c = \sigma_{ij}^n + \gamma^c D_{ijkl}\Delta\epsilon_{kl}$$

Determine the contact state for the data below

$$\begin{aligned}
&\boldsymbol{\sigma}^{nT} = [\sigma_{11} \quad \sigma_{22} \quad \sigma_{33} \quad \sigma_{21}] = [180 \quad -40 \quad 0 \quad 0] \\
&\boldsymbol{\Delta\epsilon}^T = [\Delta\epsilon_{11} \quad \Delta\epsilon_{22} \quad \Delta\epsilon_{33} \quad 2\Delta\epsilon_{21}] = [0.001 \quad 0.001 \quad 0 \quad 0] \\
&E = 210\,GPa \quad \nu = 0 \qquad \sigma_{y0} = 240 MPa
\end{aligned}$$

Hint: When $\nu = 0$ the relation $\sigma_{ij} = D_{ijkl}\epsilon_{kl}$ reduces to $\sigma_{ij} = E\epsilon_{ij}$.

16.2 Consider the yield function for an von Mises elastic-ideal plastic material material under plane stress condition

$$f = \sigma_e - \sigma_{y0} = \left[\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2\right]^{1/2} - \sigma_{y0}$$

a) Express $\dot{\lambda}$ in terms of the strain rate $\dot{\epsilon}_{ij}$ using the consistency condition $\dot{f} = 0$.
b) Integrate the the constitutive equations using one forward Euler step and one fourth-order RK step, for the scheme given in the text book, for the strain increment used in (16.1).

16.3 For the plane stress case the von Mises yield surface can be expressed in terms of principal stresses as

$$f = \sigma_e - \sigma_{y0} = \left[\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2\right]^{1/2} - \sigma_{y0}$$

Calculate the updated state for the data in task (16.1) using the backward Euler method

a) Calculate the trial stress, i.e. the updated state for elastic loading.
b) Express the the updated state in terms of $\Delta\lambda$.
c) Calculate $\Delta\lambda$ using the yield condition, $f = 0$, at the updated state.
d) Calculate the updated state, $\boldsymbol{\sigma}^{n+1}$, and compare the result with the result obtained from task (16.2).

ANSWERS

1.2

a)

$$A^T = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2.3

a)

$$\boldsymbol{\epsilon} = k \begin{bmatrix} 2x_1 & x_2 & x_3 \\ x_2 & 2x_1 & 0 \\ x_3 & 0 & 2x_1 \end{bmatrix}$$

b) Point (1,2,3) $\Rightarrow \boldsymbol{\epsilon} = k \begin{bmatrix} 2 & 2 & 3 \\ 2 & 2 & 0 \\ 3 & 0 & 2 \end{bmatrix}$

$$\boldsymbol{n}^T = \frac{1}{5}[3\ \ 4\ \ 0]\,; \qquad \boldsymbol{m}^T = \frac{1}{\sqrt{34}}[4\ \ -3\ \ 3]$$

$$\epsilon_{nn} = \frac{98}{25}k$$

$$\epsilon_{nm} = \frac{41}{5\sqrt{34}}k\,; \qquad \gamma_{nm} = 2\epsilon_{nm}$$

c) Point (1,4,0) $\Rightarrow \boldsymbol{\epsilon} = k \begin{bmatrix} 2 & 4 & 0 \\ 4 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$$\epsilon_1 = 6k \qquad \Rightarrow \qquad \boldsymbol{n}_1^T = \tfrac{1}{\sqrt{2}}[1\ 1\ 0]$$

$$\epsilon_2 = 2k \qquad \Rightarrow \qquad \boldsymbol{n}_2^T = [0\ 0\ 1]$$

$$\epsilon_3 = -2k \qquad \Rightarrow \qquad \boldsymbol{n}_3^T = \tfrac{1}{\sqrt{2}}[1\ \ -1\ 0]$$

2.4

a)

$$\epsilon' = \frac{10^{-4}}{2}\begin{bmatrix} 5+20\cos\alpha\sin\alpha & 10(\cos^2\alpha-\sin^2\alpha) & 0 \\ 10(\cos^2\alpha-\sin^2\alpha) & 5-20\cos^2\alpha\sin\alpha & 0 \\ 0 & 0 & -6 \end{bmatrix}$$

b)

$$\epsilon' = \frac{10^{-4}}{2}\begin{bmatrix} 15 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -6 \end{bmatrix}$$

i.e. a principal strain state.

2.5

a)

$$\epsilon_1 = \tfrac{10^{-4}}{2}15 \qquad \boldsymbol{n}_1^T = \tfrac{1}{\sqrt{2}}[1\ 1\ 0]$$

$$\epsilon_2 = -\tfrac{10^{-4}}{2}5 \qquad \boldsymbol{n}_2^T = \tfrac{1}{\sqrt{2}}[1\ -1\ 0]$$

$$\epsilon_3 = -\tfrac{10^{-4}}{2}6 \qquad \boldsymbol{n}_3^T = [0\ 0\ 1]$$

3.2

a) they coincide

b)

$$\boldsymbol{C} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu) \end{bmatrix}$$

3.3 The factor $\frac{1}{2}10^{-4}$ is to be excluded in the answer to **2.4**.

3.4 The factor $\frac{1}{2}10^{-4}$ is to be excluded in the answer to **2.5**.

3.5

a)

$$\boldsymbol{t} = \boldsymbol{\sigma}\boldsymbol{n} = \frac{1}{\sqrt{3}}\begin{bmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\end{bmatrix}$$

b)

$$\sigma_n = \boldsymbol{n}^T\boldsymbol{t} = \frac{1}{3}I_1$$

$$\tau_n^2 = \boldsymbol{t}^T\boldsymbol{t} - \sigma_n^2 = \frac{1}{3}(s_1^2 + s_2^2 + s_3^2) = \frac{1}{3}s_{ij}s_{ij} = \frac{2}{3}J_2$$

3.7

a)

$$\begin{aligned}
\sigma_1 &= p(1+\sqrt{3}) & \Rightarrow \quad \boldsymbol{n}_1^T &= \tfrac{1}{\sqrt{3+\sqrt{3}}}\begin{bmatrix}\tfrac{1+\sqrt{3}}{2} & \tfrac{1+\sqrt{3}}{2} & 1\end{bmatrix}\\
\sigma_2 &= 0 & \Rightarrow \quad \boldsymbol{n}_2^T &= \tfrac{1}{\sqrt{2}}\begin{bmatrix}1 & -1 & 0\end{bmatrix}\\
\sigma_3 &= p(1-\sqrt{3}) & \Rightarrow \quad \boldsymbol{n}_3^T &= \tfrac{1}{\sqrt{3-\sqrt{3}}}\begin{bmatrix}\tfrac{1-\sqrt{3}}{2} & \tfrac{1-\sqrt{3}}{2} & 1\end{bmatrix}
\end{aligned}$$

$$\boldsymbol{n}_1^T\boldsymbol{n}_2 = \boldsymbol{n}_1^T\boldsymbol{n}_3 = \boldsymbol{n}_2^T\boldsymbol{n}_3 = 0$$

b)

$$\boldsymbol{A}^T = \begin{bmatrix}\boldsymbol{n}_1 & \boldsymbol{n}_2 & \boldsymbol{n}_3\end{bmatrix} \qquad \boldsymbol{n}_1, \boldsymbol{n}_2 \text{ and } \boldsymbol{n}_3 \text{ given in a).}$$

c) Cayley-Hamilton equation: $-\boldsymbol{\sigma}^3 + \theta_1\boldsymbol{\sigma}^2 - \theta_2\boldsymbol{\sigma} - \theta_3\boldsymbol{I} = \boldsymbol{0}$ fulfilled with $\theta_1 = 2p$, $\theta_2 = -2p^2$, $\theta_3 = 0$

3.8

$$\boldsymbol{\sigma}' = \begin{bmatrix} \dfrac{\sigma_{11}+\sigma_{22}}{2}+\sigma_{12} & \dfrac{\sigma_{22}-\sigma_{11}}{2} & \dfrac{\sigma_{13}+\sigma_{23}}{\sqrt{2}} \\ \dfrac{\sigma_{22}-\sigma_{11}}{2} & \dfrac{\sigma_{11}+\sigma_{22}}{2}-\sigma_{12} & \dfrac{\sigma_{23}-\sigma_{13}}{\sqrt{2}} \\ \dfrac{\sigma_{31}+\sigma_{32}}{\sqrt{2}} & \dfrac{\sigma_{32}-\sigma_{31}}{\sqrt{2}} & \sigma_{33} \end{bmatrix}$$

3.10

$$\sigma_{11} = 2 \qquad \boldsymbol{n}^T = \frac{2}{3}\begin{bmatrix} 1 & -\frac{1}{2} & -1 \end{bmatrix}$$

4.10

a)

$$\epsilon_{ij} = \frac{\partial C}{\partial \sigma_{ij}} = (a + bI_1)s_{ij} + b\boldsymbol{J}_2\delta_{ij}$$

b) $a = \frac{3}{2}10^{-4}$ (1/MPa) $\qquad b = 10^{-5}$ (1/(MPa)^2)

c)

$$[\epsilon_{ij}] = \begin{bmatrix} b\tau^2 & a\tau & 0 \\ a\tau & b\tau^2 & 0 \\ 0 & 0 & b\tau^2 \end{bmatrix}$$

d) $\epsilon_{kk} = 3b\tau^2 = 0.3 \qquad (\epsilon_{kk})_{\text{linear elastic}} = 0$

7.1 Use $\dot{\epsilon}_{kk} = \delta_{st}\dot{\epsilon}_{st}$ and $\sigma_{mn}\dot{\epsilon}_{mn} = \sigma_{st}\dot{\epsilon}_{st}$. Moreover, use $\dot{\epsilon}_{ij} = \frac{1}{2}(\delta_{is}\delta_{jt}+\delta_{it}\delta_{js})\dot{\epsilon}_{st}$ (where also the right-hand side is symmetric in i and j). Some manipulations will show that $\dot{\sigma}_{ij} = D_{ijst}\dot{\epsilon}_{st}$ where

$$\begin{aligned} D_{ijst} = {} & \beta_1\delta_{ij}\delta_{st} + \tfrac{1}{2}\beta_2(\delta_{is}\delta_{jt} + \delta_{it}\delta_{js}) + \beta_3\sigma_{ij}\delta_{st} + \beta_4\delta_{ij}\sigma_{st} \\ & +\tfrac{1}{2}\beta_5(\sigma_{is}\delta_{jt} + \sigma_{it}\delta_{js} + \sigma_{tj}\delta_{is} + \sigma_{sj}\delta_{it}) \\ & +\beta_6\sigma_{ik}\sigma_{kj}\delta_{st} + \beta_7\sigma_{ij}\sigma_{st} + \tfrac{1}{2}\beta_8\delta_{ij}(\sigma_{tm}\sigma_{ms} + \sigma_{sm}\sigma_{mt}) \\ & +\tfrac{1}{2}\beta_9[\sigma_{ik}(\sigma_{ks}\delta_{jt} + \sigma_{kt}\delta_{js}) + \sigma_{lj}(\sigma_{tl}\delta_{is} + \sigma_{sl}\delta_{it})] \\ & +\beta_{10}\sigma_{ik}\sigma_{kj}\sigma_{st} + \tfrac{1}{2}\beta_{11}\sigma_{ij}(\sigma_{tm}\sigma_{ms} + \sigma_{sm}\sigma_{mt}) \\ & +\tfrac{1}{2}\beta_{12}\sigma_{ik}\sigma_{kj}(\sigma_{tm}\sigma_{ms} + \sigma_{sm}\sigma_{mt}) \end{aligned}$$

8.1

a) $\sigma_1 = 4$ MPa, $\boldsymbol{n}_1 = (0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$
$\sigma_2 = 2$ MPa, $\boldsymbol{n}_2 = (\frac{1}{\sqrt{2}}, -\frac{1}{2}, -\frac{1}{2})$
$\sigma_3 = 1$ MPa, $\boldsymbol{n}_3 = (\frac{1}{\sqrt{2}}, \frac{1}{2}, \frac{1}{2})$

b) s_1 =5/3 MPa, s_2 = -1/3 MPa, s_3 = -4/3 MPa

c) $J_1 = 0$, J_2 = 7/3 (MPa)^2, J_3 = 20/27 (MPa)^3

8.4 I_1 = 60 MPa, J_2 = 8 225 (MPa)^2, J_3 = 265 250 (MPa)^3, $\theta = 7.5^o$

8.5

$$\sigma = \begin{cases} 2\tau_{max} & \text{for} \quad 0 \leq \alpha \leq 1 \\ \dfrac{2}{\alpha}\tau_{max} & \text{for} \quad \alpha > 1 \\ \dfrac{2}{1-\alpha}\tau_{max} & \text{for} \quad \alpha < 0 \end{cases}$$

8.6 See Fig.8.32

8.7

a) The following (σ_1, σ_2)-points are found

1)(142.5, −22.5)	2)(−22.5, 142.5)	3)(−142.5, 22.5)	4)(22.5, −142.5)
5)(−165, −142.5)	6)(−142.5, −165)	7)(165, 142.5)	8)(142.5, 165)
9)(165, 22.5)	10)(22.5, 165)	11)(−165, −22.5)	12)(−22.5, −165)

1) from measurements, 2) change σ_1 and σ_2 (isotropy), 3) reverse the loading, 4) isotropy, 5) add hydrostatic pressure -142.5 to 1), and so on.

b) From 1) and 9) and convexity 154 MPa $\leq \sigma_{yo}$ whereas 8) and 9) and convexity gives $\sigma_{y0} \leq$ 165 MPa, i.e. 154 MPa $\leq \sigma_{y0} \leq 165$. Likewise, 2) and 3) as well as 2) and 10) gives 83 MPa $\leq \tau_{yo} \leq$ 103 MPa

c) von Mises: σ_{yo} = 155 MPa, τ_{yo} =89.5 MPa
Tresca: σ_{yo} =165 MPa, τ_{yo} =82.5 MPa

8.8 von Mises: F = 395 kN, Tresca: F = 335 kN

8.9 See Fig.8.25, $\sigma_t = \sigma_c \dfrac{1}{k}$

8.10 σ_{yo} = 216 MPa

8.12

$$a) \quad \tau_{yo} = \frac{\sigma_t \sigma_c}{\sigma_t + \sigma_c}$$

$$b) \quad \tau_{yo} = \frac{2}{\sqrt{3}} \frac{\sigma_t \sigma_c}{\sigma_t + \sigma_c}$$

$$c) \quad \tau_{yo} = \frac{\sigma_t}{\sqrt{3}}$$

$$d) \quad \tau_{yo} = \frac{\sigma_t}{2}$$

8.13 Tresca $\tau_{y0} = \sigma_{y0}/2$, von Mises $\tau_{y0} = \sigma_{y0}/\sqrt{3}$

9.2 Isotropic hardening: $\dot{\epsilon}^p_{ij} = \dot{\lambda}\frac{3s_{ij}}{2\sigma_y}$; kinematic hardening: $\dot{\epsilon}^p_{ij} = \dot{\lambda}\frac{3(s_{ij}-\alpha_{ij})}{2\sigma_{y0}}$

9.3

a) $\sigma = \sigma_{yo}/\sqrt{3}$

$$\dot{\epsilon}^p_{ij} = \dot{\lambda}\frac{\sqrt{3}}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad ; \quad \dot{\lambda} = \dot{\epsilon}^p_{eff}$$

b) $\sigma = \sigma_{yo}/2$

$$\dot{\epsilon}^p_{ij} = \dot{\lambda}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad ; \quad \dot{\lambda} = \frac{\sqrt{3}}{2}\dot{\epsilon}^p_{eff}$$

c) $\sigma = \beta/(3\alpha + \sqrt{3})$

$$\dot{\epsilon}^p_{ij} = \dot{\lambda}\begin{bmatrix} \frac{\sqrt{3}}{2}+\alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & -\frac{\sqrt{3}}{2}+\alpha \end{bmatrix} \quad ; \quad \dot{\lambda} = \frac{\dot{\epsilon}^p_{eff}}{\sqrt{1+2\alpha^2}}$$

d) $\sigma = \sigma_{yo}/(2\text{ k})$

$$\dot{\epsilon}^p_{ij} = \dot{\lambda}\begin{bmatrix} k & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad ; \quad \dot{\lambda} = \frac{\dot{\epsilon}^p_{eff}}{\sqrt{\frac{2}{3}(k^2+1)}}$$

9.4

a) $\sigma = \sigma_{yo}$

$$\dot{\epsilon}^p_{ij} = \dot{\lambda} \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad ; \quad \dot{\lambda} = \dot{\epsilon}^p_{eff}$$

b) $\sigma = \beta/(1{+}2\alpha)$

$$\dot{\epsilon}^p_{ij} = \dot{\lambda} \begin{bmatrix} \frac{1}{2} + \alpha & 0 & 0 \\ 0 & \frac{1}{2} + \alpha & 0 \\ 0 & 0 & -1 + \alpha \end{bmatrix} \quad ; \quad \dot{\lambda} = \frac{\dot{\epsilon}^p_{eff}}{\sqrt{1 + 2\alpha^2}}$$

For the criteria of Tresca and Coulomb, we obtain the situation shown in Fig.9.30. This is dealt with by *Koiter's flow rule*, not treated in the course.

9.6 Everywhere $\epsilon^p_{13} = \epsilon^p_{23} = 0$ and $\epsilon^p_{22} = \epsilon^p_{33} = -\epsilon^p_{11}/2$.
Irrespective of the load path, at point C

$$\epsilon^e_{11} = \frac{\sigma_{y0}}{E} \; ; \quad \epsilon^e_{12} = \frac{\sigma_{y0}}{2\sqrt{3}G}$$
$$\epsilon^e_{22} = \epsilon^e_{33} = -\nu\epsilon^e_{11} \; ; \quad \epsilon^e_{13} = \epsilon^e_{23} = 0$$

a) Load path AB is purely elastic; plasticity is intiated at point B.
Load path B → C

$$\epsilon^p_{11} = \frac{\sigma_{y0}}{H}(1 - \frac{\pi}{4}) \; ; \quad \epsilon^p_{12} = \frac{\sigma_{y0}}{H}\frac{\sqrt{3}}{4}\ln 2$$

b) Load path AD is purely elastic; plasticity is initiated at point D.
Load path D → C

$$\epsilon^p_{11} = \frac{\sigma_{y0}}{H}\frac{1}{2}\ln 2 \; ; \quad \epsilon^p_{12} = \frac{\sigma_{y0}}{H}\frac{\sqrt{3}}{2}(1 - \frac{\pi}{4})$$

c) Proportional loading

$$\sigma = k\sigma_{y0} \; ; \quad \tau = \frac{1}{\sqrt{3}}k\sigma_{y0}$$

where $0 \le k \le 1$. Yielding is initiated when $k = 1/\sqrt{2}$, i.e. plasticity occurs when $1/\sqrt{2} \le k \le 1$.

$$\epsilon^p_{11} = \frac{\sigma_{y0}}{H}(1 - \frac{1}{\sqrt{2}}) \; ; \quad \epsilon^p_{12} = \frac{\sigma_{y0}}{H}\frac{\sqrt{3}}{2}(1 - \frac{1}{\sqrt{2}})$$

d) The same curve $\sigma_{eff} = \sigma_{eff}(\epsilon^p_{eff})$ is obtained for all the load cases a), b) and c).